THE TWO WEIGHT $T1$ THEOREM FOR FRACTIONAL RIESZ TRANSFORMS WHEN ONE MEASURE IS SUPPORTED ON A CURVE

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Abstract. Let σ and ω be locally finite positive Borel measures on $\mathbb{R}^n$. We assume that at least one of the two measures σ and ω is supported on a regular $C^{1,\delta}$ curve in $\mathbb{R}^n$. Let $\mathbf{R}^{\alpha,n}$ be the α-fractional Riesz transform vector on $\mathbb{R}^n$. We prove the $T1$ theorem for $\mathbf{R}^{\alpha,n}$: namely that $\mathbf{R}^{\alpha,n}$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$ *if and only if* the $\mathcal{A}_2^\alpha$ conditions with holes hold, the punctured A_2^α conditions hold, and the cube testing condition for $\mathbf{R}^{\alpha,n}$ and its dual both hold. The special case of the Cauchy transform, $n=2$ and $\alpha=1$, when the curve is a line or circle, was established by Lacey, Sawyer, Shen, Uriarte-Tuero and Wick in [LaSaShUrWi].

This $T1$ theorem represents essentially the most general $T1$ theorem obtainable by methods of energy reversal. More precisely, for the pushforwards of the measures σ and ω, under a change of variable to straighten out the curve to a line, we use reversal of energy to prove that the quasienergy conditions in [SaShUr5] are implied by the $\mathcal{A}_2^\alpha$ with holes, punctured A_2^α, and quasicube testing conditions for $\mathbf{R}^{\alpha,n}$. Then we apply the main theorem in [SaShUr5] to deduce the $T1$ theorem above.

Contents

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1. Introduction

1.1. **A brief history of the $T1$ theorem.** The celebrated $T1$ theorem of David and Journé [DaJo] characterizes those singular integral operators T on $\mathbb{R}^n$ that are bounded on $L^2(\mathbb{R}^n)$, and does so in terms of a weak boundedness property, and the membership of the two functions $T\mathbf{1}$ and $T^*\mathbf{1}$ in the space of bounded mean oscillation,

$$\begin{aligned} \|T\mathbf{1}\|_{BMO(\mathbb{R}^n)} &\lesssim \|\mathbf{1}\|_{L^\infty(\mathbb{R}^n)} = 1, \\ \|T^*\mathbf{1}\|_{BMO(\mathbb{R}^n)} &\lesssim \|\mathbf{1}\|_{L^\infty(\mathbb{R}^n)} = 1. \end{aligned}$$

These latter conditions are actually the following *testing conditions* in disguise,

$$\begin{aligned} \|T\mathbf{1}_Q\|_{L^2(\mathbb{R}^n)} &\lesssim \|\mathbf{1}_Q\|_{L^2(\mathbb{R}^n)} = \sqrt{|Q|}\ , \\ \|T^*\mathbf{1}_Q\|_{L^2(\mathbb{R}^n)} &\lesssim \|\mathbf{1}_Q\|_{L^2(\mathbb{R}^n)} = \sqrt{|Q|}\ , \end{aligned}$$

tested over all indicators of cubes Q in $\mathbb{R}^n$ for both T and its dual operator T^*. This theorem was the culmination of decades of investigation into the nature of cancellation conditions required for boundedness of singular integrals[1].

A parallel thread of investigation culminated in the theorem of Coifman and Fefferman[2] that characterizes those nonnegative weights w on $\mathbb{R}^n$ for which all of the 'nicest' of the $L^2(\mathbb{R}^n)$ bounded singular integrals T above are bounded on weighted spaces $L^2(\mathbb{R}^n; w)$, and does so in terms of the A_2 condition of Muckenhoupt,

$$\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \frac{1}{w(x)}dx\right) \lesssim 1\ ,$$

taken over all cubes Q in $\mathbb{R}^n$. This condition is also a testing condition in disguise, namely it is a consequence of

$$\left\|T\left(\mathbf{s}_Q\frac{1}{w}\right)\right\|_{L^2(\mathbb{R}^n;w)} \lesssim \|\mathbf{s}_Q\|_{L^2\left(\mathbb{R}^n;\frac{1}{w}\right)}\ ,$$

tested over all 'indicators with tails' $\mathbf{s}_Q(x) = \frac{\ell(Q)}{\ell(Q)+|x-c_Q|}$ of cubes Q in $\mathbb{R}^n$.

A natural synthesis of these two results leads to the 'two weight' question of which pairs of weights (σ, ω) have the property that nice singular integrals are bounded from $L^2(\mathbb{R}^n;\sigma)$ to $L^2(\mathbb{R}^n;\omega)$. The simplest (nontrivial) singular integral of all is the Hilbert transform $Hf(x) = \int_{\mathbb{R}} \frac{f(y)}{y-x}dy$ on the real line, and Nazarov, Treil and Volberg formulated the two weight question for the Hilbert transform [Vol], that in turn led to the NTV conjecture:

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[1] See e.g. chapter VII of Stein [Ste] and the references given there for a historical background.

[2] See e.g. chapter V of [Ste] and the references given there for the long history of this investigation, in which the celebrated theorem of Hunt, Muckenhoupt and Wheeden played a critical role.

Conjecture 1. [Vol] *The Hilbert transform is bounded from $L^2(\mathbb{R}^n;\sigma)$ to $L^2(\mathbb{R}^n;\omega)$, i.e.*

$$\left\|H(f\sigma)\right\|_{L^2(\mathbb{R}^n;\omega)} \lesssim \left\|f\right\|_{L^2(\mathbb{R}^n;\sigma)}, \qquad f\in L^2(\mathbb{R}^n;\sigma),$$

if and only if the two weight A_2 condition with tails holds,

$$\left(\frac{1}{|Q|}\int_Q \mathbf{s}_Q^2 d\omega(x)\right)\left(\frac{1}{|Q|}\int_Q \mathbf{s}_Q^2 d\sigma(x)\right)\lesssim 1\ ,$$

for all cubes Q, and the two testing conditions hold,

$$\begin{aligned}\left\|H\mathbf{1}_Q\sigma\right\|_{L^2(\mathbb{R}^n;\omega)} &\lesssim \left\|\mathbf{1}_Q\right\|_{L^2(\mathbb{R}^n;\sigma)}=\sqrt{|Q|_\sigma}\ ,\\ \left\|H^*\mathbf{1}_Q\omega\right\|_{L^2(\mathbb{R}^n;\sigma)} &\lesssim \left\|\mathbf{1}_Q\right\|_{L^2(\mathbb{R}^n;\omega)}=\sqrt{|Q|_\omega}\ ,\end{aligned}$$

for all cubes Q.

In a groundbreaking series of papers including [NTV1],[NTV2] and [NTV4], Nazarov, Treil and Volberg used weighted Haar decompositions with random grids, introduced their 'pivotal' condition, and proved the above conjecture under the side assumption that the pivotal condition held. Subsequently, in joint work of two of us, Sawyer and Uriarte-Tuero, with Lacey [LaSaUr2], it was shown that the pivotal condition was not necessary in general, a necessary 'energy' condition was introduced as a substitute, and a hybrid merging of these two conditions was shown to be sufficient for use as a side condition. Eventually, these three authors with Shen established the NTV conjecture in a two part paper; Lacey, Sawyer, Shen and Uriarte-Tuero [LaSaShUr3] and Lacey [Lac]. A key ingredient in the proof was an 'energy reversal' phenomenom enabled by the Hilbert transform kernel equality

$$\frac{1}{y-x}-\frac{1}{y-x'}=\frac{x-x'}{(y-x)(y-x')},$$

having the remarkable property that the denominator on the right hand side remains *positive* for all y outside the smallest interval containing both x and x'. This proof of the NTV conjecture was given in the special case that the weights σ and ω had no point masses in common, largely to avoid what were then thought to be technical issues. However, these issues turned out to be considerably more interesting, and this final assumption of no common point masses was removed shortly after by Hytönen [Hyt2], who also simplified some aspects of the proof.

At this juncture, attention naturally turned to the analogous two weight inequalities for higher dimensional singular integrals, as well as α-fractional singular integrals such as the Cauchy transform in the plane. In a long paper [SaShUr4], begun on the *arXiv* in 2013, the authors introduced the appropriate notions of Poisson kernel to deal with the A_2^α condition on the one hand, and the α-energy condition on the other hand (unlike for the Hilbert transform, these two Poisson kernels differ in general). The main result of that paper established the $T1$ theorem for 'elliptic' vectors of singular integrals under the side assumption that an energy condition and its dual held, thus identifying the *culprit* in higher dimensions as the energy conditions (see also [SaShUr5] where the restriction to no common point masses was removed). A general $T1$ conjecture is this.

Conjecture 2. *Let $\mathbf{T}^{\alpha,n}$ denote an elliptic vector of standard α-fractional singular integrals in $\mathbb{R}^n$. Then $\mathbf{T}^{\alpha,n}$ is bounded from $L^2(\mathbb{R}^n;\sigma)$ to $L^2(\mathbb{R}^n;\omega)$, i.e.*

$$\left\|\mathbf{T}^{\alpha,n}(f\sigma)\right\|_{L^2(\mathbb{R}^n;\omega)} \lesssim \left\|f\right\|_{L^2(\mathbb{R}^n;\sigma)}, \qquad f\in L^2(\mathbb{R}^n;\sigma),$$

if and only if the two one-tailed $\mathcal{A}_2^\alpha$ conditions with holes hold, the punctured A_2^α conditions hold, and the two testing conditions hold,

$$\begin{aligned}\left\|\mathbf{T}^{\alpha,n}\mathbf{1}_Q\sigma\right\|_{L^2(\mathbb{R}^n;\omega)} &\lesssim \left\|\mathbf{1}_Q\right\|_{L^2(\mathbb{R}^n;\sigma)}=\sqrt{|Q|_\sigma}\ ,\\ \left\|\mathbf{T}^{\alpha,n,\mathrm{dual}}\mathbf{1}_Q\omega\right\|_{L^2(\mathbb{R}^n;\sigma)} &\lesssim \left\|\mathbf{1}_Q\right\|_{L^2(\mathbb{R}^n;\omega)}=\sqrt{|Q|_\omega}\ ,\end{aligned}$$

for all cubes Q in $\mathbb{R}^n$ (whose sides need not be parallel to the coordinate axes).

A positive resolution to this conjecture could have implications for a number of problems that are higher dimensional analogues of those connected to the Hilbert transform (see e.g. [Vol], [NiTr], [NaVo], [VoYu], [PeVoYu], [PeVoYu1], [IwMa], [LaSaUr], [AsGo] and [AsZi]).

In view of the aforementioned main result in [SaShUr5], the following conjecture is stronger.

Conjecture 3. *Let $\mathbf{T}^{\alpha,n}$ denote an elliptic vector of standard α-fractional singular integrals in $\mathbb{R}^n$. If $\mathbf{T}^{\alpha,n}$ is bounded from $L^2(\mathbb{R}^n;\sigma)$ to $L^2(\mathbb{R}^n;\omega)$, then the energy conditions hold as defined in Definition 23 below.*

At the time of this writing, it is not known if the energy conditions are necessary or not in general. The paper [LaWi2] by Lacey and Wick overlaps [SaShUr4] to some extent.

While no counterexamples have yet been discovered to the energy conditions, there are some cases in which they have been proved to hold. Of course, the energy conditions hold for the Hilbert transform on the line [LaSaUr2], and in recent joint work with M. Lacey and B. Wick, the five of us have established that the energy conditions hold for the Cauchy transform in the plane in the special case where one of the measures is supported on either a straight line or a circle, thus proving the $T1$ theorem in this case. The key to this result was an extension of the energy reversal phenomenon for the Hilbert transform to the setting of the Cauchy transform, and here the one-dimensional nature of the line and circle played a critical role. In particular, a special decomposition of a 2-dimensional measure into 'end' and 'side' pieces played a crucial role, and was in fact discovered independently in both the initial version of this paper and in [LaWi].

In this paper, we extend the $T1$ theorem to the setting where one of the measures is supported on a Hölder continuously differentiable curve (in higher dimensions). This result seems to represent the best possible $T1$ theorem that can be obtained from the methods of energy reversal, and requires a number of new ideas, especially of a geometric nature, as opposed to the more algebraic 'corona' ideas developed for the solution to the NTV conjecture. In particular, changes of variable are made to straighten sufficiently small pieces of the curve to a line, and the resulting operator norms, A_2^α conditions, and testing condition constants are tracked under these changes of variable. This tracking presents significant subtleties, especially for the testing constants, which require appropriate tangent plane approximations to the phase function of the testing kernel. Further effort is then needed to control the testing conditions associated with these pieces by the testing conditions we assume for the curve in the first place. In particular, a 'localized triple testing' condition is derived that enables the reduction of testing conditions for small pieces of transformed measure on a line to the testing conditions for the global measures. Yet another complication arises here in the use of quasicubes in the proof - dictated by

pushforwards of ordinary cubes - and this requires the new notion of 'L-transverse' and its properties in order to control the intersections of quasicubes with lines.

We now give a more precise description of what is in this paper and its relation to the literature.

1.2. **Statement of results.** In [SaShUr5] (see also [SaShUr] and [SaShUr4] for special cases), under a side assumption that certain *energy conditions* hold, the authors show in particular that the two weight inequality

$$\left\|\mathbf{R}^{\alpha,n}\left(f\sigma\right)\right\|_{L^2(\omega)} \lesssim \left\|f\right\|_{L^2(\sigma)}, \tag{1.1}$$

for the vector of Riesz transforms $\mathbf{R}^{\alpha,n}$ in $\mathbb{R}^n$ (with $0 \leq \alpha < n$) holds if and only if the $\mathcal{A}_2^\alpha$ conditions with holes hold, the punctured $A_2^{\alpha,\mathrm{punct}}$ conditions hold, the quasicube testing conditions hold, and the quasiweak boundedness property holds. Here a quasicube is a globally biLipschitz image of a usual cube. Precise definitions of all terms used here are given in the next section. It is not known at the time of this writing whether or not these or any other energy conditions are necessary for *any* vector $\mathbf{T}^{\alpha,n}$ of fractional singular integrals in $\mathbb{R}^n$ with $n \geq 2$, apart from the trivial case of positive operators. In particular there are no known counterexamples. We also showed in [SaShUr2] and [SaShUr3] that the technique of reversing energy, typically used to prove energy conditions, fails spectacularly in higher dimension (and we thank M. Lacey for showing us this failure for the Cauchy transform with the circle measure). See also the counterexamples for the fractional Riesz transforms in [LaWi2].

The purpose of this paper is to show that if σ and ω are locally finite positive Borel measures (possibly having common point masses), and at least *one* of the two measures σ and ω is supported on a line in $\mathbb{R}^n$, or on a regular $C^{1,\delta}$ curve in $\mathbb{R}^n$, then the energy conditions are indeed necessary for boundedness of the fractional Riesz transform $\mathbf{R}^{\alpha,n}$, and hence that a $T1$ theorem holds for $\mathbf{R}^{\alpha,n}$ (see Theorem 9 below). Just after the first version of this paper appeared on the *arXiv*, M. Lacey and B. Wick [LaWi] independently posted a similar result for the special case of the Cauchy transform in the plane where one measure is supported on a line or a circle, and the five authors have combined on the paper [LaSaShUrWi] in this setting.

The vector of α-fractional Riesz transforms is given by

$$\mathbf{R}^{\alpha,n} = \left\{R_\ell^{\alpha,n} : 1 \leq \ell \leq n\right\}, \qquad 0 \leq \alpha < n,$$

where the component Riesz transforms $R_\ell^{\alpha,n}$ are the convolution fractional singular integrals $R_\ell^{\alpha,n} f \equiv K_\ell^{\alpha,n} * f$ with odd kernel defined by

$$K_\ell^{\alpha,n}\left(w\right) \equiv c_{\alpha,n} \frac{w^\ell}{\left|w\right|^{n+1-\alpha}}.$$

Finally, we remark that the $T1$ theorem under this geometric condition has application to the weighted discrete Hilbert transform $H_{(\Gamma,v)}$ when the sequence Γ is supported on an appropriate $C^{1,\delta}$ curve in the complex plane. See [BeMeSe] where $H_{(\Gamma,v)}$ is essentially the Cauchy transform with $n = 2$ and $\alpha = 1$.

We now recall a special case of our main two weight theorem from [SaShUr5] which plays a key role here - see also [SaShUr] and [SaShUr4] for earlier versions. Let $\mathcal{P}^n$ denote the collection of all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes, and denote by $\mathcal{D}^n \subset \mathcal{P}^n$ a dyadic grid in $\mathbb{R}^n$. The side conditions $\mathcal{A}_2^\alpha$, $\mathcal{A}_2^{\alpha,\mathrm{dual}}$,

$A_2^{\alpha,\mathrm{punct}}$, $A_2^{\alpha,\mathrm{punct},\mathrm{dual}}$, $\mathcal{E}_\alpha$ and $\mathcal{E}_\alpha^{\mathrm{dual}}$ depend only only on the measure pair (σ,ω), while the necessary conditions $\mathfrak{T}_{\mathbf{R}^{\alpha,n}}$, $\mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\mathrm{dual}}$ and $\mathcal{WBP}_{\mathbf{R}^{\alpha,n}}$ depend on the measure pair (σ,ω) as well as the singular operator $\mathbf{R}_\sigma^{\alpha,n}$. These conditions will be explained below. For convenience in notation, we use Fraktur font for A to denote,

$$\mathfrak{A}_2^\alpha \equiv \mathcal{A}_2^\alpha + \mathcal{A}_2^{\alpha,\mathrm{dual}} + A_2^{\alpha,\mathrm{punct}} + A_2^{\alpha,\mathrm{punct},\mathrm{dual}},$$

or when the measure pair is important,

$$\mathfrak{A}_2^\alpha(\sigma,\omega) \equiv \mathcal{A}_2^\alpha(\sigma,\omega) + \mathcal{A}_2^{\alpha,\mathrm{dual}}(\sigma,\omega) + A_2^{\alpha,\mathrm{punct}}(\sigma,\omega) + A_2^{\alpha,\mathrm{punct},\mathrm{dual}}(\sigma,\omega). \tag{1.2}$$

Notation 4. *In order to avoid confusion with the use of $*$ for pullbacks and push-forwards of maps, we will use the superscript* dual *in place of $*$ to denote 'dual conditions' throughout this paper.*

Theorem 5. *Suppose that $\mathbf{R}^{\alpha,n}$ is the vector of α-fractional Riesz transforms in $\mathbb{R}^n$, and that ω and σ are positive locally finite Borel measures on $\mathbb{R}^n$ (possibly having common point masses). Set $\mathbf{R}_\sigma^{\alpha,n} f = \mathbf{R}^{\alpha,n}(f\sigma)$ for any smooth truncation of $\mathbf{R}^{\alpha,n}$. Let $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ be a globally biLipschitz map.*

(1) *Suppose $0 \le \alpha < n$ and that $\gamma \ge 2$ is given. Then the vector Riesz transform $\mathbf{R}_\sigma^{\alpha,n}$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$, i.e.*

$$\left\| \mathbf{R}_\sigma^{\alpha,n} f \right\|_{L^2(\omega)} \le \mathfrak{N}_{\mathbf{R}^{\alpha,n}} \| f \|_{L^2(\sigma)}, \tag{1.3}$$

uniformly in smooth truncations of $\mathbf{R}^{\alpha,n}$, and moreover

$$\mathfrak{N}_{\mathbf{R}^{\alpha,n}} \le C_\alpha \left(\sqrt{\mathfrak{A}_2^\alpha} + \mathcal{E}_\alpha + \mathcal{E}_\alpha^{\mathrm{dual}} + \mathfrak{T}_{\mathbf{R}^{\alpha,n}} + \mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\mathrm{dual}} + \mathcal{WBP}_{\mathbf{R}^{\alpha,n}} \right), \tag{1.4}$$

provided that the two dual $\mathcal{A}_2^\alpha$ conditions with holes hold, the punctured dual $A_2^{\alpha,\mathrm{punct}}$ conditions hold, and the two dual quasicube testing conditions for $\mathbf{R}^{\alpha,n}$ hold, the quasiweak boundedness property for $\mathbf{R}^{\alpha,n}$ holds for a sufficiently large constant C depending on the goodness parameter $\mathbf{r}$, and provided that the two dual quasienergy conditions $\mathcal{E}_\alpha + \mathcal{E}_\alpha^{\mathrm{dual}} < \infty$ hold uniformly over all dyadic grids $\mathcal{D}^n$, and where the goodness parameters $\mathbf{r}$ and ε implicit in the definition of $\mathcal{M}_{\mathbf{r}-\mathrm{deep}}^\ell(K)$ below are fixed sufficiently large and small respectively depending on n, α and γ. Here $\mathfrak{N}_{\mathbf{R}^{\alpha,n}} = \mathfrak{N}_{\mathbf{R}^{\alpha,n}}(\sigma,\omega)$ is the least constant in (1.3).

(2) *Conversely, suppose $0 \le \alpha < n$ and that the Riesz transform vector $\mathbf{R}_\sigma^{\alpha,n}$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$,*

$$\left\| \mathbf{R}_\sigma^{\alpha,n} f \right\|_{L^2(\omega)} \le \mathfrak{N}_{\mathbf{R}^{\alpha,n}} \| f \|_{L^2(\sigma)}.$$

Then the testing conditions and weak boundedness property hold for $\mathbf{R}^{\alpha,n}$, the fractional $\mathcal{A}_2^\alpha$ conditions with holes hold, and the punctured dual $A_2^{\alpha,\mathrm{punct}}$ conditions hold, and moreover,

$$\sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{\mathbf{R}^{\alpha,n}} + \mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\mathrm{dual}} + \mathcal{WBP}_{\mathbf{R}^{\alpha,n}} \le C \mathfrak{N}_{\mathbf{R}^{\alpha,n}}.$$

Problem 6. *It is an open question whether or not the energy conditions are necessary for boundedness of $\mathbf{R}_\sigma^{\alpha,n}$. See* [SaShUr3] *for a failure of* energy reversal *in higher dimensions - such an energy reversal was used in dimension $n=1$ to prove the necessity of the energy condition for the Hilbert transform.*

Remark 7. *In* [LaWi2]*, M. Lacey and B. Wick use the NTV technique of surgery to show that an* expectation *over grids of an analogue of the weak boundedness property for the Riesz transform vector $\mathbf{R}^{\alpha,n}$ is controlled by the $\mathcal{A}_2^\alpha$ and cube testing*

conditions, together with a small multiple of the operator norm. They then claim a $T1$ theorem with a side condition of uniformly full dimensional measures, using independent grids corresponding to each measure, resulting in an elimination of the weak boundedness property as a condition. In any event, the weak boundedness property is always necessary for the norm inequality, and as such can be viewed as a weak close cousin of the testing conditions.

The main result of this paper is the $T1$ theorem for a measure supported on a regular $C^{1,\delta}$ curve. We point out that the cubes occurring in the testing conditions in the following theorem are of course restricted to those that intersect the curve, otherwise the integrals vanish, and include not only those in $\mathcal{P}^n$ with sides parallel to the axes, but also those in $\mathcal{Q}^n$ consisting of all rotations of the cubes in $\mathcal{P}^n$:

$$\begin{aligned} \mathfrak{T}_{\mathbf{R}^{\alpha,n}}^2 &\equiv \sup_{Q\in\mathcal{Q}^n} \frac{1}{|Q|_\sigma} \int_Q \left|\mathbf{R}^{\alpha,n}\left(\mathbf{1}_Q\sigma\right)\right|^2 \omega < \infty, \\ \left(\mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\mathrm{dual}}\right)^2 &\equiv \sup_{Q\in\mathcal{Q}^n} \frac{1}{|Q|_\omega} \int_Q \left|\left(\mathbf{R}^{\alpha,n}\right)^{\mathrm{dual}}\left(\mathbf{1}_Q\omega\right)\right|^2 \sigma < \infty. \end{aligned} \tag{1.5}$$

In the special case considered in [LaSaShUrWi] of the Cauchy transform in the plane with ω supported on the unit circle $\mathbb{T}$ or the real line $\mathbb{R}$, the testing is taken over the smaller collection of all Carleson squares.

We consider *regular* $C^{1,\delta}$ curves in $\mathbb{R}^n$ defined as follows.

Definition 8. *Suppose $\delta>0$, $I=[a,b]$ is a closed interval on the real line with $-\infty<a<b<\infty$, and that $\Phi:I\to\mathbb{R}^n$ is a $C^{1,\delta}$ curve parameterized by arc length. The curve is one-to-one with the possible exception that $\Phi(a)=\Phi(b)$. We refer to any curve as above as a* regular *$C^{1,\delta}$ curve.*

Theorem 9. *Let $0\leq\alpha<n$ and suppose Φ is a regular $C^{1,\delta}$ curve. Suppose further that*

(1) σ and ω are positive locally finite Borel measures on $\mathbb{R}^n$ (possibly having common point masses), and ω is supported in $\mathcal{L}\equiv\mathrm{range}\,\Phi$, and

(2) $\mathbf{R}^{\alpha,n}$ is the vector of α-fractional Riesz transforms in $\mathbb{R}^n$, and $\mathbf{R}_\sigma^{\alpha,n}f=\mathbf{R}^{\alpha,n}(f\sigma)$ for any smooth truncation of $\mathbf{R}^{\alpha,n}$.

Then $\mathbf{R}_\sigma^{\alpha,n}$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$, i.e.

$$\left\|\mathbf{R}_\sigma^{\alpha,n}f\right\|_{L^2(\omega)} \leq \mathfrak{N}_{\mathbf{R}^{\alpha,n}} \|f\|_{L^2(\sigma)},$$

uniformly in smooth truncations of $\mathbf{R}^{\alpha,n}$, if and only if *the two dual $\mathcal{A}_2^\alpha$ conditions with holes hold, the punctured dual $A_2^{\alpha,\mathrm{punct}}$ conditions hold, and the two dual cube testing conditions (1.5) for $\mathbf{R}^{\alpha,n}$ hold. Moreover we have the equivalence*

$$\mathfrak{N}_{\mathbf{R}^{\alpha,n}} \approx \sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{\mathbf{R}^{\alpha,n}} + \mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\mathrm{dual}}.$$

1.3. **Techniques.** The remainder of the introduction is devoted to giving an overview of the techniques and arguments needed to obtain Theorem 9 from Theorem 5. For this we need Ω-quasicubes and conformal α-fractional Riesz transforms $\mathbf{R}_\Psi^{\alpha,n}$ where Ω is a globally biLipschitz map and Ψ is a $C^{1,\delta}$ diffeomorphism of $\mathbb{R}^n$. We now describe these issues in more detail. Let $\Omega:\mathbb{R}^n\to\mathbb{R}^n$ be a globally biLipschitz

map as defined in Definition 19 below, and refer to the images ΩQ of cubes in $\mathcal{Q}^n$ under the map Ω as Ω-quasicubes or simply quasicubes. These Ω-quasicubes will often be used in place of cubes in the testing conditions, energy conditions and weak boundedness property, and we will use $\Omega\mathcal{Q}^n$ as a superscript to indicate this. For example, the quasitesting analogue of the usual testing conditions (1.5) is:

$$\begin{aligned}(1.6)\qquad \left(\mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\Omega\mathcal{Q}^n}\right)^2 &\equiv \sup_{Q\in\Omega\mathcal{Q}^n} \frac{1}{|Q|_\sigma}\int_Q \left|\mathbf{R}^{\alpha,n}\left(\mathbf{1}_Q\sigma\right)\right|^2\omega<\infty,\\ \left(\mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{dual}}\right)^2 &\equiv \sup_{Q\in\Omega\mathcal{Q}^n} \frac{1}{|Q|_\omega}\int_Q \left|\left(\mathbf{R}^{\alpha,n}\right)^{\mathrm{dual}}\left(\mathbf{1}_Q\omega\right)\right|^2\sigma<\infty.\end{aligned}$$

We alert the reader to the fact that different collections of quasicubes will be considered in the course of proving our theorem. The definitions of these terms, and the remaining terms used below, will be given precisely in the next section. When the superscript $\Omega\mathcal{Q}^n$ is omitted, it is understood that the quasicubes are the usual cubes $\mathcal{Q}^n$.

The next result shows that the quasienergy conditions are in fact necessary for boundedness of the Riesz transform vector $\mathbf{R}^{\alpha,n}$ when one of the measures is supported on a *line*. In that case, the quasienergy conditions are even implied by the Muckenhoupt $\mathcal{A}_2^\alpha$ conditions with holes and the quasitesting conditions. Moreover, the backward tripled quasitesting condition and the quasiweak boundedness property are implied by the Muckenhoupt with holes and quasitesting conditions as well, but provided the quasicubes come from a C^1 diffeomorphism and are rotated in an appropriate way.

Finally, in order to obtain the $T1$ theorem when one measure is supported on a curve, we will need to generalize the fractional Riesz transforms $\mathbf{R}^{\alpha,n}$ that we can consider in this theorem. Consider $\Psi:\mathbb{R}^n\to\mathbb{R}^n$ given by

$$\Psi(x)=\left(x^1,x^2-\psi^2\left(x^1\right),x^3-\psi^3\left(x^1\right),...,x^n-\psi^n\left(x^1\right)\right)=x-\left(0,\psi\left(x_1\right)\right),$$

where $x=\left(x^1,x'\right)$ and

$$\psi(t)=\left(\psi^2(t),\psi^3(t),...,\psi^n(t)\right)\in\mathbb{R}^{n-1},$$

is a $C^{1,\delta}$ function $\psi:\mathbb{R}\to\mathbb{R}^{n-1}$. Let $\mathbf{K}^{\alpha,n}(x,y)$ denote the vector Riesz kernel and define

$$\mathbf{K}_\Psi^{\alpha,n}(x,y)=\frac{|y-x|^{n+1-\alpha}}{|\Psi(y)-\Psi(x)|^{n+1-\alpha}}\mathbf{K}^{\alpha,n}(x,y)=c_{\alpha,n}\frac{y-x}{|\Psi(y)-\Psi(x)|^{n+1-\alpha}}.$$

Definition 10. *We refer to the operator* $\mathbf{R}_\Psi^{\alpha,n}$ *with kernel* $\mathbf{K}_\Psi^{\alpha,n}$ *as a* conformal *α-fractional Riesz transform. We also define the factor*

$$\Gamma_\Psi(x,y)\equiv\frac{|y-x|^{n+1-\alpha}}{|\Psi(y)-\Psi(x)|^{n+1-\alpha}}$$

to be the conformal factor associated with Ψ *and* $\mathbf{K}^{\alpha,n}$.

Notation 11. *We emphasize that the* $C^{1,\delta}$ *diffeomorphism* Ψ *that appears in the definition of the conformal α-fractional Riesz transform* $\mathbf{R}_\Psi^{\alpha,n}$ *need not have any relation to the biLipschitz map* $\Omega:\mathbb{R}^n\to\mathbb{R}^n$ *that is used to define the quasicubes under consideration. On the other hand, we will have reason to consider* Ψ*-quasicubes as well in connection with changes of variable.*

We use the tangent line approximations to $\mathbf{R}_{\Psi}^{\alpha,n}$ having kernels $\mathbf{K}_{\Psi}^{\alpha,n}(x,y)\,\rho_{\eta,R}^{\alpha}$ where $\rho_{\eta,R}^{\alpha}$ is defined in the next section. It is shown in [SaShUr5] (see also [LaSaShUr3] for the one-dimensional case without holes) that one can replace these tangent line truncations with any reasonable notion of truncation, including the usual cutoff truncations.

We now introduce a condition on Ω-quasicubes that plays a role in deriving the necessity of the tripled testing and weak boundedness conditions. See Lemma 26 below for the relevant consequences of this condition. We begin with a collection of 'good' rotations R that take the standard basis $\{\mathbf{e}_i\}_{i=1}^n$ to a basis $\{R\mathbf{e}_j\}_{j=1}^n$ in which no unit vector $R\mathbf{e}_j$ is too close to any unit vector $\mathbf{e}_i$.

Let $\mathfrak{R}^n$ denote the group of rotations in $\mathbb{R}^n$. For $\mathbf{e}\in\mathbb{S}^{n-1}$ and $0<\eta<1$ let

$$\digamma_{\mathbf{e},\eta}\equiv\left\{R\in\mathfrak{R}^n:\left|\langle R\mathbf{e},\mathbf{e}_k\rangle\right|\leq\eta\text{ for }1\leq k\leq n\right\}.$$

Note that the condition $\digamma_{\mathbf{e},\eta}\neq\emptyset$ is independent of the unit vector $\mathbf{e}$, and depends only on η, by transitivity of rotations. Fix $\eta=\eta_n\in(0,1)$ so that $\digamma_{\mathbf{e},\eta}\neq\emptyset$ for all $\mathbf{e}\in\mathbb{S}^{n-1}$ (this requires $\eta_n\geq\frac{1}{\sqrt{n}}$).

Definition 12. *Let L be a line in $\mathbb{R}^n$. A C^1 diffeomorphism $\Omega:\mathbb{R}^n\to\mathbb{R}^n$ is* L-transverse *if*

$$\left\|D\Omega^{-1}-R\right\|_{\infty}<\frac{1-\eta}{4}$$

for some $R\in\digamma_{\mathbf{e}_L,\eta}$ where $\mathbf{e}_L$ is a unit vector in the direction of L.

Theorem 13. *Fix a collection of Ω-quasicubes. Let σ and ω be locally finite positive Borel measures on $\mathbb{R}^n$ (possibly having common point masses). Suppose that $\mathbf{R}_{\Psi}^{\alpha,n}$ is a conformal α-fractional Riesz transform with $0\leq\alpha<n$, and where Ψ is a $C^{1,\delta}$ diffeomorphism with $\Psi(x)=x-(0,\psi(x_1))$ where*

$$\|D\psi\|_{\infty}<\frac{1}{8n}\left(1-\frac{\alpha}{n}\right). \tag{1.7}$$

Impose the tangent line truncations for $\mathbf{R}_{\Psi}^{\alpha,n}$ in the Ω-quasitesting conditions. If the measure ω is supported on a line L, then

$$\mathcal{E}_{\alpha}^{\Omega\mathcal{Q}^n}\lesssim\sqrt{\mathcal{A}_2^{\alpha}}+\mathfrak{T}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Omega\mathcal{Q}^n}\text{ and }\mathcal{E}_{\alpha}^{\Omega\mathcal{Q}^n,\mathrm{dual}}\lesssim\sqrt{\mathcal{A}_2^{\alpha,\mathrm{dual}}}+\mathfrak{T}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{dual}}.$$

If in addition Ω is a C^1 diffeomorphism and L-transverse, then

$$\mathcal{WBP}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Omega\mathcal{P}^n}\lesssim\mathfrak{T}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Omega\mathcal{P}^n,\mathrm{triple},\mathrm{dual}}\lesssim\mathfrak{T}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Omega\mathcal{P}^n,\mathrm{dual}}+\sqrt{\mathcal{A}_2^{\alpha}}+\sqrt{\mathcal{A}_2^{\alpha,\mathrm{dual}}}.$$

Remark 14. *We restrict Theorem 13 to conformal **Riesz** transforms $\mathbf{R}_{\Psi}^{\alpha,n}$ in order to exploit the special property that for $j\geq2$, the scalar transforms $R_j^{\alpha,n}$ and $\left(R_{\Psi}^{\alpha,n}\right)_j$ behave like a Poisson operator when acting on a measure supported on the x_1-axis. This property is not shared by higher order Riesz transforms, such as the Beurling transform in the plane, and this accounts for our failure to treat such singular integrals at this time. The restriction to **conformal** Riesz transforms $\mathbf{R}_{\Psi}^{\alpha,n}$ is dictated by the reversal of energy that is possible for these special transforms when the phase of the singular integral is $y-x$ and one of the measures is supported on a line.*

Since the conformal factor $\Gamma_\Psi(x,y)$ in Definition 10 satisfies the estimates

$$
\begin{aligned}
\frac{1}{C} &\leq \Gamma(x,y) \leq C, \qquad (1.8)\\
|\nabla\Gamma(x,y)| &\leq C|x-y|^{-1},\\
|\nabla\Gamma(x,y)-\nabla\Gamma(x',y)| &\leq C\left(\frac{|x-x'|}{|x-y|}\right)^{\delta}|x-y|^{-1}, \qquad \frac{|x-x'|}{|x-y|}\leq\frac{1}{2},\\
|\nabla\Gamma(x,y)-\nabla\Gamma(x,y')| &\leq C\left(\frac{|y-y'|}{|x-y|}\right)^{\delta}|x-y|^{-1}, \qquad \frac{|y-y'|}{|x-y|}\leq\frac{1}{2},
\end{aligned}
$$

it is easy to see from the product rule that the conformal fractional Riesz transforms are standard fractional singular integrals in the sense used in [SaShUr5]. Since they are also strongly elliptic as in [SaShUr5], it follows from the main theorem in [SaShUr5] that

Conclusion 15. *Theorem 5 above holds with $\mathbf{R}_\Psi^{\alpha,n}$ in place of $\mathbf{R}^{\alpha,n}$ provided Ψ is a $C^{1,\delta}$ diffeomorphism.*

If we combine Theorem 13 with this extension of Theorem 5, we immediately obtain the following $T1$ theorem as a corollary.

Remark 16. *The following theorem generalizes the $T1$ theorem for the Hilbert transform (*[Lac], [LaSaShUr3] *and* [Hyt2]*) both in that the supports of measures are more general, and in that the kernels treated are more general. See also related work in the references given at the end of the paper.*

Theorem 17. *Fix a line L and a collection of Ω-quasicubes and suppose that Ω is a C^1 diffeomorphism and L-transverse. Let σ and ω be locally finite positive Borel measures on $\mathbb{R}^n$ (possibly having common point masses). Suppose that $\mathbf{R}_\Psi^{\alpha,n}$ is a conformal fractional Riesz transform with $0\leq\alpha<n$, where Ψ is a $C^{1,\delta}$ diffeomorphism given by $\Psi(x)=x-(0,\psi(x_1))$ where ψ satisfies (1.7), i.e.*

$$\|D\psi\|_\infty < \frac{1}{8n}\left(1-\frac{\alpha}{n}\right).$$

Set $(\mathbf{R}_\Psi^{\alpha,n})_\sigma f=\mathbf{R}_\Psi^{\alpha,n}(f\sigma)$ for any smooth truncation of $\mathbf{R}_\Psi^{\alpha,n}$. If at least one of the measures σ and ω is supported on the line L, then the operator norm $\mathfrak{N}_{\mathbf{R}_\Psi^{\alpha,n}}$ of $(\mathbf{R}_\Psi^{\alpha,n})_\sigma$ as an operator from $L^2(\sigma)$ to $L^2(\omega)$, uniformly in smooth truncations, satisfies

$$\mathfrak{N}_{\mathbf{R}_\Psi^{\alpha,n}} \approx C_\alpha\left(\sqrt{\mathfrak{A}_2^\alpha}+\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n}+\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{dual}}\right).$$

Our extension of Theorem 17 to the case when one measure is compactly supported on a $C^{1,\delta}$ curve $\mathcal{L}$ presented as a graph requires additional work. More precisely, we suppose that $\mathcal{L}$ is presented as the graph of a $C^{1,\delta}$ function $\psi:\mathbb{R}\to\mathbb{R}^{n-1}$ given by

$$\psi(t)=\left(\psi^2(t),\psi^3(t),...,\psi^n(t)\right)\in\mathbb{R}^{n-1}, \qquad t\in\mathbb{R}.$$

Define $\Psi:\mathbb{R}^n\to\mathbb{R}^n$ by

$$\Psi(x)=\left(x^1,x^2-\psi^2(x^1),x^3-\psi^3(x^1),...,x^n-\psi^n(x^1)\right)=x-\left(0,\psi(x^1)\right),$$

where $x=\left(x^1,x'\right)$. Then Ψ is globally invertible with inverse map

$$\Psi^{-1}(\xi)=\left(\xi^1,\xi^2+\psi^2(\xi_1),\xi^3+\psi^3(\xi_1),...,\xi^n+\psi^n(\xi_1)\right)=\xi+(0,\psi(\xi_1)).$$

Both Ψ and its inverse Ψ^{-1} are $C^{1,\delta}$ maps, and $\Psi\mid_{\mathcal{L}}$ is a $C^{1,\delta}$ diffeomorphism from the curve $\mathcal{L}$ to the x_1-axis. Set $\Psi_*\mathcal{Q}^n = \left(\Psi^{-1}\right)^*\mathcal{Q}^n = \{\Psi Q : Q \in \mathcal{Q}^n\}$. The images ΨQ of cubes Q under the map Ψ are Ψ-*quasicubes.*

The next theorem is a preliminary version of the main Theorem 9 that requires only the change of variable estimates in Propositions 28 and 29 below. The 'defects' in this preliminary version are that the quasitesting conditions are related to the map Ψ defining the curve $\mathcal{L}$, and that the smallness condition (1.7) is imposed on the derivative of ψ.

Theorem 18. *Let $n \geq 2$ and $0 \leq \alpha < n$. Suppose that $\mathcal{L}$ is a $C^{1,\delta}$ curve in $\mathbb{R}^n$ presented as the graph of a $C^{1,\delta}$ function $\psi : \mathbb{R} \to \mathbb{R}^{n-1}$ as above, and assume that (1.7) holds, i.e.*

$$\|D\psi\|_\infty < \frac{1}{8n}\left(1-\frac{\alpha}{n}\right).$$

Let ω and σ be positive Borel measures (possibly having common point masses), and assume that ω is compactly supported in $\mathcal{L}$. Let Ψ be associated to ψ as above. Finally, set $\mathcal{R}^n = R\mathcal{P}^n$ where R is a rotation that is L-transverse when L is the x_1-axis. Then the α-fractional Riesz transform $\mathbf{R}^{\alpha,n}$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$ if and only if the Muckenhoupt conditions hold, $\mathcal{A}_2^\alpha + \mathcal{A}_2^{\alpha,\mathrm{dual}} + A_2^{\alpha,\mathrm{punct}} + A_2^{\alpha,\mathrm{punct,dual}} < \infty$, and the quasitesting conditions hold, $\mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\Psi\mathcal{R}^n} + \mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\Psi\mathcal{R}^n,\mathrm{dual}} < \infty$, where $\mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\Psi\mathcal{R}^n}$ and $\mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\Psi\mathcal{R}^n,\mathrm{dual}}$ are the best constants in

$$\begin{aligned}
&\int_{\Psi Q}\left|\mathbf{R}^{\alpha,n}\left(\mathbf{1}_{\Psi Q}\sigma\right)\right|^2 d\omega \leq \left(\mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\Psi\mathcal{R}^n}\right)^2 |\Psi Q|_\sigma\,,\\
&\int_{\Psi Q}\left|\mathbf{R}^{\alpha,n,\mathrm{dual}}\left(\mathbf{1}_{\Psi Q}\omega\right)\right|^2 d\sigma \leq \left(\mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\Psi\mathcal{R}^n,\mathrm{dual}}\right)^2 |\Psi Q|_\omega\,,\\
&\qquad\text{for all cubes } Q \in \mathcal{R}^n = R\mathcal{P}^n.
\end{aligned}$$

Moreover we have the equivalence

$$\mathfrak{N}_{\mathbf{R}_\sigma^{\alpha,n}}(\sigma,\omega) \approx \sqrt{\mathfrak{A}_2^\alpha(\sigma,\omega)} + \mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\Psi\mathcal{R}^n}(\sigma,\omega) + \mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\Psi\mathcal{R}^n,\mathrm{dual}}(\sigma,\omega).$$

The bound on $\|D\psi\|_\infty$ can be relaxed, but we will not pursue this here. To obtain Theorem 9, we instead remove the Lipschitz assumption by cutting the support $\mathcal{L}$ of ω into sufficiently small pieces $\mathcal{L}_i$ where the oscillation of the tangents to $\mathcal{L}_i$ is small. Here the necessity of the tripled testing condition $\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{triple,dual}}$ in Theorem 13 plays a key role in permitting our testing conditions to be taken with respect to the entire measure ω, rather than with respect to the corresponding pieces $\mathbf{1}_{\mathcal{L}_i}\omega$. Then the restriction to *quasi*testing conditions is removed using the fact that Theorem 5 holds for conformal Riesz transforms with general quasicubes, see Conclusion 15.

Finally, we mention another direction in which Theorem 17 can be generalized, namely to the setting where σ and ω are locally finite positive Borel measures supported on a (k_1+1)-dimensional subspace S and a (k_2+1)-dimensional subspace W respectively of $\mathbb{R}^n$, $n = k_1 + k_2 + 1$, with W and S intersecting at right angles in a line L. The precise result and its proof are given in the appendix at the end of the paper.

2. Definitions

The α-fractional *Riesz* vector $\mathbf{R}^{\alpha,n} = \{R_\ell^{\alpha,n} : 1 \leq \ell \leq n\}$ has as components the Riesz transforms $R_\ell^{n,\alpha}$ with odd kernel $K_\ell^{\alpha,n}(w) = \frac{\Omega_\ell(w)}{|w|^{n-\alpha}}$. The *tangent line truncation* of the Riesz transform $R_\ell^{\alpha,n}$ has kernel $\Omega_\ell(w)\,\rho_{\eta,R}^{\alpha}(|w|)$ where $\rho_{\eta,R}^{\alpha}$ is continuously differentiable on an interval $(0,S)$ with $0<\eta<R<S$, and where $\rho_{\eta,R}^{\alpha}(r) = r^{\alpha-n}$ if $\eta \leq r \leq R$, and has constant derivative on both $(0,\eta)$ and (R,S) where $\rho_{\eta,R}^{\alpha}(S) = 0$. As shown in [SaShUr5] (see [LaSaShUr3] for the one dimensional case without holes), boundedness of $R_\ell^{n,\alpha}$ with one set of appropriate truncations together with the offset A_2^α condition (see below), is equivalent to boundedness of $R_\ell^{n,\alpha}$ with all truncations. In particular this includes the smooth truncations with kernels $\varphi_{\eta,R}(|w|)\,K_\ell^{\alpha,n}(w) = \varphi_{\eta,R}(|w|)\frac{\Omega_\ell(w)}{|w|^{n-\alpha}}$ where $\varphi_{\eta,R}$ is infinitely differentiable and compactly supported on the interval $(0,\infty)$ with $0<\eta<R<\infty$, and where $\varphi_{\eta,R}(r) = 1$ if $\eta \leq r \leq R$.

2.1. Quasicubes.

Our general notion of quasicube will be derived from the following definition.

Definition 19. *We say that a map $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ is a* globally biLipschitz map *if*

$$\|\Omega\|_{Lip} \equiv \sup_{\substack{x,y\in\mathbb{R}^n\\ x\neq y}} \frac{\|\Omega(x)-\Omega(y)\|}{\|x-y\|} < \infty,$$

and $\|\Omega^{-1}\|_{Lip} < \infty$. We say that a map $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is a $C^{1,\delta}$ diffeomorphism *if*

$$\|\Psi\|_{C^{1,\delta}} \equiv \sup_{x\in\mathbb{R}^n} \|\nabla\Psi(x)\| + \sup_{\substack{x,y\in\mathbb{R}^n\\ x\neq y}} \frac{\|\nabla\Psi(x)-\nabla\Psi(y)\|}{\|x-y\|^\delta} < \infty,$$

and $\|\Psi^{-1}\|_{C^{1,\delta}} < \infty$. When $\delta = 0$, we write $C^1 = C^{1,0}$ and $\|\Psi\|_{C^1} \equiv \sup_{x\in\mathbb{R}^n}\|\nabla\Psi(x)\|$.

Note that if Ω is a globally biLipschitz map, then there are constants $c, C > 0$ such that

$$c \leq J_\Omega(x) \equiv |\det D\Omega(x)| \leq C, \qquad x\in\mathbb{R}^n.$$

Special cases of globally biLipschitz maps are given by $C^{1,\delta}$ diffeomorphisms $\Psi : \mathbb{R}^n \to \mathbb{R}^n$, and these include those used in the definition of conformal Riesz transforms above, and defined by

$$\Psi(x) = x - \left(0, \psi\left(x^1\right)\right),$$

where $x = \left(x^1, x'\right) \in \mathbb{R}^n$ and $\psi : \mathbb{R} \to \mathbb{R}^{n-1}$ is a $C^{1,\delta}$ function. We denote by $\mathcal{Q}^n$ the collection of *all* cubes in $\mathbb{R}^n$, and by $\mathcal{P}^n$ the subcollection of cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes, and by $\mathcal{D}^n$ (contained in $\mathcal{P}^n$) a dyadic grid in $\mathbb{R}^n$.

Definition 20. *Suppose that $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ is a globally biLipschitz map.*

(1) *If E is a measurable subset of $\mathbb{R}^n$, we define $\Omega E \equiv \{\Omega(x) : x \in E\}$ to be the image of E under the homeomorphism Ω.*
   (a) *In the special case that $E = Q \in \mathcal{Q}^n$ is a cube in $\mathbb{R}^n$, we will refer to ΩQ as a quasicube (or Ω-quasicube if Ω is not clear from the context).*
   (b) *We define the center of the quasicube ΩQ to be Ωx_Q where x_Q is the center of Q.*

   (c) *We define the side length* $\ell(\Omega Q)$ *of the quasicube* ΩQ *to be the side-length* $\ell(Q)$ *of the cube* Q.
   (d) *For* $r>0$ *we define the 'dilation'* $r\Omega Q$ *of a quasicube* ΩQ *to be* ΩrQ *where* rQ *is the usual 'dilation' of a cube in* $\mathbb{R}^n$ *that is concentric with* Q *and having side length* $r\ell(\Omega Q)$.
(2) *If* $\mathcal{K}$ *is a collection of cubes in* $\mathbb{R}^n$, *we define* $\Omega\mathcal{K}\equiv\{\Omega Q : Q\in\mathcal{K}\}$ *to be the collection of quasicubes* ΩQ *as* Q *ranges over* $\mathcal{K}$.
(3) *If* $\mathcal{F}$ *is a grid of cubes in* $\mathbb{R}^n$, *we define the inherited grid structure on* $\Omega\mathcal{F}$ *by declaring that* ΩQ *is a child of* $\Omega Q'$ *in* $\Omega\mathcal{F}$ *if* Q *is a child of* Q' *in the grid* $\mathcal{F}$.

Note that if ΩQ is a quasicube, then $|\Omega Q|^{\frac{1}{n}}\approx|Q|^{\frac{1}{n}}=\ell(Q)=\ell(\Omega Q)$ shows that the measure of ΩQ is approximately its sidelength to the power n. Moreover, there is a positive constant R_{big} such that we have the comparability containments

$$Q+\Omega x_Q\subset R_{\mathrm{big}}\Omega Q \text{ and } \Omega Q\subset R_{\mathrm{big}}(Q+\Omega x_Q)\ . \tag{2.1}$$

2.2. **The $\mathcal{A}_2^\alpha$ conditions.** Recall that $\Omega:\mathbb{R}^n\to\mathbb{R}^n$ is a globally biLipschitz map. Now let μ be a locally finite positive Borel measure on $\mathbb{R}^n$, and suppose Q is a Ω-quasicube in $\mathbb{R}^n$. Recall that $|Q|^{\frac{1}{n}}\approx\ell(Q)$ for a quasicube Q. The two α-fractional Poisson integrals of μ on a quasicube Q are given by:

$$\begin{aligned}
\mathrm{P}^\alpha(Q,\mu) &\equiv \int_{\mathbb{R}^n}\frac{|Q|^{\frac{1}{n}}}{\left(|Q|^{\frac{1}{n}}+|x-x_Q|\right)^{n+1-\alpha}}d\mu(x)\,,\\
\mathcal{P}^\alpha(Q,\mu) &\equiv \int_{\mathbb{R}^n}\left(\frac{|Q|^{\frac{1}{n}}}{\left(|Q|^{\frac{1}{n}}+|x-x_Q|\right)^{2}}\right)^{n-\alpha}d\mu(x)\,,
\end{aligned}$$

where we emphasize that $|x-x_Q|$ denotes Euclidean distance between x and x_Q and $|Q|$ denotes the Lebesgue measure of the quasicube Q. We refer to P^α as the *standard* Poisson integral and to $\mathcal{P}^\alpha$ as the *reproducing* Poisson integral. Let σ and ω be locally finite positive Borel measures on $\mathbb{R}^n$, possibly having common point masses, and suppose $0\leq\alpha<n$.

We say that the pair (K,K') in $\mathcal{Q}^n\times\mathcal{Q}^n$ are *neighbours* if K and K' live in a common dyadic grid and both $K\subset 3K'\setminus K'$ and $K'\subset 3K\setminus K$, and we denote by $\mathcal{N}^n$ the set of pairs (K,K') in $\mathcal{Q}^n\times\mathcal{Q}^n$ that are neighbours. Let $\Omega\mathcal{N}^n=\Omega\mathcal{Q}^n\times\Omega\mathcal{Q}^n$ be the corresponding collection of neighbour pairs of quasicubes. Then we define the classical *offset* A_2^α *constants* by

$$A_2^\alpha(\sigma,\omega)\equiv\sup_{(Q,Q')\in\Omega\mathcal{N}^n}\frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}\frac{|Q'|_\omega}{|Q|^{1-\frac{\alpha}{n}}}.$$

Since the cubes in $\mathcal{P}^n$ are products of half open, half closed intervals $[a,b)$, the neighbouring quasicubes $(Q,Q')\in\Omega\mathcal{N}^n$ are disjoint, and the common point masses of σ nor ω do simultaneously appear in each factor.

We now define the *one-tailed* $\mathcal{A}_2^\alpha$ constant using $\mathcal{P}^\alpha$. The energy constants $\mathcal{E}_\alpha$ introduced in the next subsection will use the standard Poisson integral P^α.

Definition 21. *The one-sided constants* $\mathcal{A}_2^\alpha$ *and* $\mathcal{A}_2^{\alpha,\mathrm{dual}}$ *for the weight pair* (σ,ω) *are given by*

$$\begin{aligned}\mathcal{A}_2^\alpha(\sigma,\omega) &\equiv \sup_{Q\in\Omega\mathcal{Q}^n}\mathcal{P}^\alpha\left(Q,\mathbf{1}_{Q^c}\sigma\right)\frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}}<\infty,\\ \mathcal{A}_2^{\alpha,\mathrm{dual}}(\sigma,\omega) &\equiv \sup_{Q\in\Omega\mathcal{Q}^n}\frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}\mathcal{P}^\alpha\left(Q,\mathbf{1}_{Q^c}\omega\right)<\infty.\end{aligned}$$

Note that these definitions are the analogues of the corresponding conditions with ‘holes’ introduced by Hytönen [Hyt] in dimension $n=1$ - the supports of the measures $\mathbf{1}_{Q^c}\sigma$ and $\mathbf{1}_Q\omega$ in the definition of $\mathcal{A}_2^\alpha$ are disjoint, and so the common point masses of σ and ω do not appear simultaneously in each factor.

2.2.1. *Punctured Muckenhoupt conditions.* Given an at most countable set $\mathfrak{P}=\{p_k\}_{k=1}^\infty$ in $\mathbb{R}^n$, a quasicube $Q\in\Omega\mathcal{Q}^n$, and a positive locally finite Borel measure μ, define

$$\mu(Q,\mathfrak{P})\equiv|Q|_\mu-\sup\left\{\mu(p_k):p_k\in Q\cap\mathfrak{P}\right\},$$

where we note that the sup above is achieved at some point p_k since μ is locally finite. The quantity $\mu(Q,\mathfrak{P})$ is simply the $\widetilde{\mu}$ measure of Q where $\widetilde{\mu}$ is the measure μ with its largest point mass in Q removed. Given a locally finite measure pair (σ,ω), let $\mathfrak{P}_{(\sigma,\omega)}=\{p_k\}_{k=1}^\infty$ be the at most countable set of common point masses of σ and ω. Then as shown in [SaShUr5] (as pointed out by Hytönen [Hyt2], the one-dimensional case follows from the proof of Proposition 2.1 in [LaSaUr2]), the weighted norm inequality (1.1) implies finiteness of the following *punctured* Muckenhoupt conditions:

$$\begin{aligned}A_2^{\alpha,\mathrm{punct}}(\sigma,\omega) &\equiv \sup_{Q\in\Omega\mathcal{Q}^n}\frac{\sigma\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)}{|Q|^{1-\frac{\alpha}{n}}}\frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}},\\ A_2^{\alpha,\mathrm{punct,dual}}(\sigma,\omega) &\equiv \sup_{Q\in\Omega\mathcal{Q}^n}\frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}\frac{\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)}{|Q|^{1-\frac{\alpha}{n}}}.\end{aligned}$$

Finally, we point out that the intersection of these conditions, namely $\mathcal{A}_2^\alpha+\mathcal{A}_2^{\alpha,\mathrm{dual}}A_2^{\alpha,\mathrm{punct}}+A_2^{\alpha,\mathrm{punct,dual}}<\infty$, is independent of the biLipschitz map Ω as follows from taking $\Psi=\Omega^{-1}$ in Proposition 28 below.

2.3. **Quasicube testing and quasiweak boundedness property.** The following ‘dual’ quasicube testing conditions are necessary for the boundedness of $\mathbf{R}^{\alpha,n}$ from $L^2(\sigma)$ to $L^2(\omega)$:

$$\begin{aligned}\mathfrak{T}_{\mathbf{R}^{\alpha,n}}^2 &\equiv \sup_{Q\in\Omega\mathcal{Q}^n}\frac{1}{|Q|_\sigma}\int_Q\left|\mathbf{R}^{\alpha,n}\left(\mathbf{1}_Q\sigma\right)\right|^2\omega<\infty,\\ \left(\mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\mathrm{dual}}\right)^2 &\equiv \sup_{Q\in\Omega\mathcal{Q}^n}\frac{1}{|Q|_\omega}\int_Q\left|\left(\mathbf{R}^{\alpha,n}\right)^{\mathrm{dual}}\left(\mathbf{1}_Q\omega\right)\right|^2\sigma<\infty.\end{aligned}$$

Note that these conditions are required to hold uniformly over tangent line truncations of $\mathbf{R}^{\alpha,n}$, and where again we point out that in the presence of the $\mathcal{A}_2^\alpha$ conditions, we can equivalently replace the tangent line truncations with any other admissible truncations.

The quasiweak boundedness property for $\mathbf{R}^{\alpha,n}$ is another necessary condition for (1.1) given by

$$\left|\int_Q \mathbf{R}^{\alpha,n}\left(1_{Q'}\sigma\right)d\omega\right| \leq \mathcal{WBP}_{\mathbf{R}^{\alpha,n}}\sqrt{\left|Q\right|_\omega\left|Q'\right|_\sigma},$$

$$\text{for all dyadic quasicubes } Q,Q'\in\Omega\mathcal{D} \text{ with } \frac{1}{C}\leq\frac{\left|Q\right|^{\frac{1}{n}}}{\left|Q'\right|^{\frac{1}{n}}}\leq C,$$

$$\text{and either } Q\subset 3Q'\setminus Q' \text{ or } Q'\subset 3Q\setminus Q,$$

$$\text{and all dyadic quasigrids } \Omega\mathcal{D}.$$

2.4. **Quasienergy conditions.** Suppose $\Omega:\mathbb{R}^n\to\mathbb{R}^n$ is a $C^{1,\delta}$ diffeomorphism. We begin by briefly recalling some of the notation used in [SaShUr5]. Given a dyadic Ω-quasicube $K\in\mathcal{D}$ and a positive measure μ we define the Ω-quasiHaar projection $\mathsf{P}_K^\mu\equiv\sum_{J\in\mathcal{D}:\ J\subset K}\triangle_J^\mu$ where the projections $\triangle_J^\mu$ are the usual orthogonal projections onto the space of mean value zero functions that are constant on the children of J - see e.g. [SaShUr5]. Now we recall the definition of a *good* dyadic quasicube - see [NTV4] and [LaSaUr2] and [SaShUr] for more detail - and the definition of a quasicube that is *deeply embedded* in another quasicube. We say that a dyadic quasicube J is $(\mathbf{r},\varepsilon)$-*deeply embedded* in a dyadic quasicube K, or simply $\mathbf{r}$-*deeply embedded* in K, which we write as $J\Subset_{\mathbf{r}}K$, when $J\subset K$ and both

$$\begin{aligned}\ell\left(J\right) &\leq 2^{-\mathbf{r}}\ell\left(K\right),\\ \operatorname{quasidist}\left(J,\partial K\right) &\geq \frac{1}{2}\ell\left(J\right)^\varepsilon\ell\left(K\right)^{1-\varepsilon},\end{aligned}\tag{2.2}$$

where we define the quasidistance $\operatorname{quasidist}\left(E,F\right)$ between two sets E and F to be the Euclidean distance $\operatorname{dist}\left(\Omega^{-1}E,\Omega^{-1}F\right)$ between the preimages $\Omega^{-1}E$ and $\Omega^{-1}F$ of E and F under the map Ω, and where we recall that $\ell\left(J\right)\approx\left|J\right|^{\frac{1}{n}}$.

Definition 22. *Let $\mathbf{r}\in\mathbb{N}$ and $0<\varepsilon<1$. A dyadic quasicube J is $(\mathbf{r},\varepsilon)$-*good*, or simply* good*, if for* every *dyadic superquasicube I, it is the case that **either** J has side length at least $2^{-\mathbf{r}}$ times that of I, **or** $J\Subset_{\mathbf{r}}I$ is $(\mathbf{r},\varepsilon)$-deeply embedded in I.*

The parameters $\mathbf{r},\varepsilon$ will be fixed sufficiently large and small respectively later on, and we denote the set of such good dyadic quasicubes by $\Omega\mathcal{D}_{\text{good}}$. We thus have

$$\left\|\mathsf{P}_I^\mu\mathbf{x}\right\|_{L^2(\mu)}^2=\int_I\left|\mathbf{x}-\mathbb{E}_I^\mu\mathbf{x}\right|^2d\mu\left(x\right)=\int_I\left|\mathbf{x}-\left(\frac{1}{\left|I\right|_\mu}\int_I\mathbf{x}dx\right)\right|^2d\mu\left(x\right),\quad \mathbf{x}=\left(x_1,...,x_n\right),$$

where $\mathsf{P}_I^\mu\mathbf{x}$ is the orthogonal projection of the identity function $\mathbf{x}:\mathbb{R}^n\to\mathbb{R}^n$ onto the vector-valued subspace of $\oplus_{k=1}^n L^2\left(\mu\right)$ consisting of functions supported in I with μ-mean value zero, and where $\mathbb{E}_I^\mu\mathbf{x}$ is the expectation (μ-average) of $\mathbf{x}$ on the cube I. At this point we emphasize that in the setting of quasicubes we continue to use the linear function $\mathbf{x}$ and not the pushforward of $\mathbf{x}$ by Ω. The reason of course is that the quasienergy defined below is used to capture the first order information in the Taylor expansion of a singular kernel.

We use the collection $\mathcal{M}_{\mathbf{r}-\text{deep}}\left(K\right)$ of *maximal* $\mathbf{r}$-deeply embedded dyadic subquasicubes of a dyadic quasicube K. We let $J^*=\gamma J$ where $\gamma\geq 2$. The goodness parameter $\mathbf{r}$ is chosen sufficiently large, depending on ε and γ, that the bounded

overlap property

$$\sum_{J\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}(K)}\mathbf{1}_{J^*}\leq\beta\mathbf{1}_K\ , \tag{2.3}$$

holds for some positive constant β depending only on $n,\gamma,\mathbf{r}$ and ε (see [SaShUr4]). We will also need the following refinements of $\mathcal{M}_{\mathbf{r}-\mathrm{deep}}(K)$ for each $\ell\geq 0$:

$$\mathcal{M}^{\ell}_{\mathbf{r}-\mathrm{deep}}(K)\equiv\left\{J\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}\left(\pi^{\ell}K\right):J\subset L\text{ for some }L\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}(K)\right\},$$

where $\pi^{\ell}K$ denotes the ℓ^{th} parent above K in the dyadic grid. Since $J\in\mathcal{M}^{\ell}_{\mathbf{r}-\mathrm{deep}}(K)$ implies $\gamma J\subset K$, we also have from (2.3) that

$$\sum_{J\in\mathcal{M}^{(\ell)}_{\mathbf{r}-\mathrm{deep}}(K)}\mathbf{1}_{J^*}\leq\beta\mathbf{1}_K\ ,\qquad\text{for each }\ell\geq 0. \tag{2.4}$$

Of course $\mathcal{M}^{0}_{\mathbf{r}-\mathrm{deep}}(K)=\mathcal{M}_{\mathbf{r}-\mathrm{deep}}(K)$, but $\mathcal{M}^{\ell}_{\mathbf{r}-\mathrm{deep}}(K)$ is in general a finer subdecomposition of K the larger ℓ is, and may in fact be empty.

There is one final generalization we need. We say that a quasicube $J\in\Omega\mathcal{P}^n$ is $(\mathbf{r},\varepsilon)$-*deeply embedded* in a quasicube $K\in\Omega\mathcal{P}^n$, or simply $\mathbf{r}$-*deeply embedded* in K, which we write as $J\Subset_{\mathbf{r}}K$, when $J\subset K$ and both

$$\begin{aligned}\ell(J)&\leq 2^{-\mathbf{r}}\ell(K),\\ \mathrm{quasidist}(J,\partial K)&\geq\frac{1}{2}\ell(J)^{\varepsilon}\ell(K)^{1-\varepsilon}.\end{aligned}$$

This is the same definition as we gave earlier for *dyadic* quasicubes, but is now extended to arbitrary quasicubes $J,K\in\Omega\mathcal{P}^n$. Now given $K\in\Omega\mathcal{D}^n$ and $F\in\Omega\mathcal{P}^n$ with $\ell(F)\geq\ell(K)$, define

$$\mathcal{M}^{F}_{\mathbf{r}-\mathrm{deep}}(K)\equiv\left\{\text{maximal }J\in\Omega\mathcal{D}^n:J\Subset_{\mathbf{r}}K\text{ and }J\Subset_{\mathbf{r}}F\right\},$$

and

$$\left(\mathcal{E}^{\mathrm{xrefined}}_{\alpha}\right)^2\equiv\sup_I\sup_{F\in\Omega\mathcal{P}^n:\ \ell(F)\geq 2\ell(I)}\frac{1}{|I|_{\sigma}}\sum_{J\in\mathcal{M}^{F}_{\mathbf{r}-\mathrm{deep}}(I)}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{I\setminus\gamma J}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}^{\mathrm{subgood},\omega}_{J}\mathbf{x}\right\|^2_{L^2(\omega)}.$$

The important difference here is that the quasicube $F\in\Omega\mathcal{P}^n$ is permitted to lie outside the quasigrid $\Omega\mathcal{D}^n$ containing K. Similarly we have a dual version of $\mathcal{E}^{\mathrm{xrefined}}_{\alpha}$.

Definition 23. *Suppose σ and ω are positive Borel measures on $\mathbb{R}^n$. Then the quasienergy condition constant $\mathcal{E}^{\Omega\mathcal{Q}^n}_{\alpha}$ is given by*

$$\left(\mathcal{E}^{\Omega\mathcal{Q}^n}_{\alpha}\right)^2\equiv\sup_{\substack{F\in\Omega\mathcal{P}^n\\ \ell(F)\geq\ell(I)}}\sup_{I=\dot\cup I_r}\frac{1}{|I|_{\sigma}}\sum_{r=1}^{\infty}\sum_{J\in\mathcal{M}^{F}_{\mathbf{r}-\mathrm{deep}}(I_r)}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{I\setminus\gamma J}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}^{\omega}_{J}\mathbf{x}\right\|^2_{L^2(\omega)}\ ,$$

where $\sup_{I=\dot\cup I_r}$ above is taken over

1. *all dyadic quasigrids $\Omega\mathcal{D}$,*
2. *all $\Omega\mathcal{D}$-dyadic quasicubes I,*
3. *and all subpartitions $\{I_r\}_{r=1}^{N\ or\ \infty}$ of the quasicube I into $\Omega\mathcal{D}$-dyadic subquasicubes I_r.*

This definition of the quasienergy constant $\mathcal{E}_{\alpha}^{\Omega\mathcal{Q}^n}$ is larger than that used in [SaShUr5]. There is a similar definition for the dual (backward) quasienergy condition that simply interchanges σ and ω everywhere. These definitions of the quasienergy condition depend on the choice of goodness parameters $\mathbf{r}$ and ε.

Finally, we record the following elementary special case of the Energy Lemma (see e.g. [SaShUr5] or [SaShUr] or [LaWi]) that we will need here. Recall that our quasicubes come from a fixed globally biLipschitz map Ω in $\mathbb{R}^n$. Our singular integrals below will be conformal fractional Riesz transforms associated with an unrelated $C^{1,\delta}$ diffeomorphism Ψ of $\mathbb{R}^n$ that is presented as a graph.

Lemma 24 (**Quasienergy Lemma**). *Suppose that Ω is a globally biLipschitz map, and that Ψ is a $C^{1,\delta}$ diffeomorphism of $\mathbb{R}^n$. Let J be a quasicube in $\Omega\mathcal{D}^{\omega}$. Let ψ_J be an $L^2(\omega)$ function supported in J and with ω-integral zero. Let ν be a positive measure supported in $\mathbb{R}^n \setminus \gamma J$ with $\gamma \geq 2$. Then for $\mathbf{R}_{\Psi}^{\alpha,n}$ a conformal α-fractional Riesz transform, we have*

$$\left|\left\langle \mathbf{R}_{\Psi}^{\alpha,n}(\nu), \psi_J\right\rangle_{\omega}\right| \lesssim \left\|\psi_J\right\|_{L^2(\omega)} \left(\frac{\mathrm{P}^{\alpha}(J,\nu)}{|J|^{\frac{1}{n}}}\right) \left\|\mathsf{P}_J^{\omega}\mathbf{x}\right\|_{L^2(\omega)}.$$

3. One measure supported in a line

In this section we prove Theorem 13, i.e. we prove that the Ω-quasienergy conditions, the backward tripled Ω-quasitesting conditions (for appropriately rotated quasicubes), and the Ω-quasiweak boundedness property (for appropriately rotated quasicubes) are implied by the Muckenhoupt $\mathcal{A}_2^{\alpha}$ conditions and the Ω-quasitesting conditions $\mathfrak{T}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Omega\mathcal{Q}^n}$ and $\mathfrak{T}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{dual}}$ associated to the tangent line truncations of a conformal α-fractional Riesz transform $\mathbf{R}_{\Psi}^{\alpha,n}$, when *one* of the measures ω is supported in a certain line, and the other measure σ is arbitrary. The one-dimensional character of just one of the measures is enough to circumvent the failure of strong reversal of energy as described in [SaShUr2] and [SaShUr3].

Notation 25. *We emphasize again that the $C^{1,\delta}$ diffeomorphism Ψ that appears in the definition of the conformal α-fractional Riesz transform $\mathbf{R}_{\Psi}^{\alpha,n}$ need not have any relation to the globally biLipschitz map Ω that is used to define the quasicubes under consideration.*

Recall that the conformal Riesz transforms $\mathbf{R}_{\Psi}^{\alpha,n}$ considered here have vector kernel $\mathbf{K}_{\Psi}^{\alpha,n}$ defined by

$$\mathbf{K}_{\Psi}^{\alpha,n}(y,x) \equiv \frac{y-x}{\left|\Psi(y)-\Psi(x)\right|^{n+1-\alpha}},$$

where we suppose that Ψ is given as the graph of $\psi : \mathbb{R} \to \mathbb{R}^{n-1}$:

$$\Psi(x) = \left(x^1, x' + \psi\left(x^1\right)\right) = \left(x^1, x^2 + \psi^2\left(x^1\right), ..., x^n + \psi^n\left(x^1\right)\right), \tag{3.1}$$

where $\psi \in C^{1,\delta}$.

Fix a collection of Ω-quasicubes where $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ is a globally biLipschitz map unrelated to Ψ. Fix a dyadic quasigrid $\Omega\mathcal{D}$, and suppose that ω is supported in the x_1-axis, which we denote by L. We will show that both quasienergy conditions hold relative to $\Omega\mathcal{D}$. Furthermore, when Ω is a C^1 diffeomorphism and L-transverse, we will show that the backward tripled quasitesting condition, and hence also the quasiweak boundedness property, is controlled by $\mathcal{A}_2^{\alpha,\mathrm{dual}}$ and dual quasitesting.

Let $3 < \gamma = \gamma(n,\alpha)$ where γ will be taken sufficiently large depending on n and α for the arguments below to be valid - see in particular (3.7), (3.8), (**??**) and (3.16) below - and where we also need $\|D\psi\|_\infty$ sufficiently small depending on n and α as in (1.7) above, i.e.

$$\|D\psi\|_\infty < \frac{1}{8n^2}(n-\alpha). \tag{3.2}$$

3.1. **Backward quasienergy condition.** The dual (backward) quasienergy condition $\mathcal{E}_\alpha^{\Omega\mathcal{Q}^n,\mathrm{dual}} \lesssim \mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{dual}} + \sqrt{\mathcal{A}_2^{\alpha,\mathrm{dual}}}$ is the more straightforward of the two to verify, and so we turn to it first. We will show

$$\sup_{\ell\geq 0}\sum_{r=1}^\infty \sum_{J\in\mathcal{M}_{\mathrm{deep}}^\ell(I_r)} \left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus J^*}\omega\right)}{|J|^{\frac{1}{n}}}\right)^2 \|\mathsf{P}_J^\sigma \mathbf{x}\|_{L^2(\sigma)}^2 \leq \left(\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{dual}}\right)^2 |I|_\omega \ ,$$

for all partitions of a dyadic quasicube $I = \bigcup_{r=1}^\infty I_r$ into dyadic subquasicubes I_r. We fix $\ell \geq 0$ and suppress both ℓ and $\mathbf{r}$ in the notation $\mathcal{M}_{\mathrm{deep}}(I_r) = \mathcal{M}_{\mathbf{r}-\mathrm{deep}}^\ell(I_r)$. Recall that $J^* = \gamma J$, and that the bounded overlap property (2.4) holds. We may of course assume that I intersects the x_1-axis L. Now we set $\mathcal{M}_{\mathrm{deep}} \equiv \bigcup_{r=1}^\infty \mathcal{M}_{\mathrm{deep}}(I_r)$ and write

$$\sum_{r=1}^\infty \sum_{J\in\mathcal{M}_{\mathrm{deep}}(I_r)} \left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus\gamma J}\omega\right)}{|J|^{\frac{1}{n}}}\right)^2 \|\mathsf{P}_J^\sigma \mathbf{x}\|_{L^2(\sigma)}^2 = \sum_{J\in\mathcal{M}_{\mathrm{deep}}} \left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus\gamma J}\omega\right)}{|J|^{\frac{1}{n}}}\right)^2 \|\mathsf{P}_J^\sigma \mathbf{x}\|_{L^2(\sigma)}^2 \ .$$

We will consider the cases $3J\cap L=\emptyset$ and $3J\cap L\neq\emptyset$ separately.

Suppose $3J\cap L=\emptyset$. There is $c>0$ and a finite sequence $\{\xi_k\}_{k=1}^N$ in $\mathbb{S}^{n-1}$ (actually of the form $\xi_k = \left(0,\xi_k^2,...,\xi_k^n\right)$) with the following property. For each $J\in\mathcal{M}_{\mathrm{deep}}$ with $3J\cap L=\emptyset$, there is $1\leq k=k(J)\leq N$ such that for $y\in J$ and $x\in I\cap L$, the linear combination $\xi_k\cdot\mathbf{K}_\Psi^{\alpha,n}(y,x)$ is positive and satisfies

$$\xi_k\cdot\mathbf{K}_\Psi^{\alpha,n}(y,x) = \frac{\xi_k\cdot(y-x)}{|\Psi(y)-\Psi(x)|^{n+1-\alpha}} \gtrsim \frac{\ell(J)}{|y-x|^{n+1-\alpha}}.$$

For example, in the plane $n=2$, if J lies above the x_1-axis L, then for $y\in J$ and $x\in L$ we have $y_2 \gtrsim (3-1)\ell(J) > \ell(J)$ and $x_2=0$, hence the estimate

$$(0,1)\cdot\mathbf{K}_\Psi^{\alpha,n}(y,x) = \frac{y_2-x_2}{|\Psi(y)-\Psi(x)|^{n+1-\alpha}} \gtrsim \frac{\ell(J)}{|y-x|^{n+1-\alpha}}.$$

For J below L we take the unit vector $(0,-1)$ in place of $(0,1)$. Thus for $y\in J\in\mathcal{M}_{\mathrm{deep}}$ and $k=k(J)$ we have the following 'weak reversal' of quasienergy for the conformal Riesz transform $\mathbf{R}_\Psi^{\alpha,n}$ with kernel $\mathbf{K}_\Psi^{\alpha,n}(y,x)$,

$$\begin{aligned} \left|\mathbf{R}_\Psi^{\alpha,n}(\mathbf{1}_{I\cap L}\omega)(y)\right| &= \left|\int_{I\cap L}\mathbf{K}_\Psi^{\alpha,n}(y,x)\,d\omega(x)\right| \\ &\geq \left|\int_{I\cap L}\xi_k\cdot\mathbf{K}_\Psi^{\alpha,n}(y,x)\,d\omega(x)\right| \\ &\gtrsim \int_{I\cap L}\frac{\ell(J)}{|y-x|^{n+1-\alpha}}d\omega(x) \approx \mathrm{P}^\alpha(J,\mathbf{1}_I\omega). \end{aligned} \tag{3.3}$$

Thus from (3.3) and the pairwise disjointedness of $J\in\mathcal{M}_{\mathrm{deep}}$, we have

$$\begin{aligned}
&\sum_{\substack{J\in\mathcal{M}_{\mathrm{deep}}\\ 3J\cap L=\emptyset}}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{I}\omega\right)}{\left|J\right|^{\frac{1}{n}}}\right)^{2}\left\|\mathsf{P}_{J}^{\sigma}\mathbf{x}\right\|_{L^{2}(\sigma)}^{2}\lesssim\sum_{\substack{J\in\mathcal{M}_{\mathrm{deep}}\\ 3J\cap L=\emptyset}}\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{I}\omega\right)^{2}\left|J\right|_{\sigma}\\
\lesssim\;&\sum_{J\in\mathcal{M}_{\mathrm{deep}}}\int_{J}\left|\mathbf{R}_{\Psi}^{\alpha,n}\left(\mathbf{1}_{I\cap L}\omega\right)(y)\right|^{2}d\sigma(y)\\
\leq\;&\int_{I}\left|\mathbf{R}_{\Psi}^{\alpha,n}\left(\mathbf{1}_{I}\omega\right)(y)\right|^{2}d\sigma(y)\leq\left(\mathfrak{T}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Omega\mathcal{Q}^{n},\mathrm{dual}}\right)^{2}\left|I\right|_{\omega}\ .
\end{aligned}$$

Now we turn to estimating the sum over those quasicubes $J\in\mathcal{M}_{\mathrm{deep}}$ for which $3J\cap L\neq\emptyset$. In this case we use the one-dimensional nature of the support of ω to obtain a strong reversal of one of the partial quasienergies. Recall the Hilbert transform inequality for intervals J and I with $2J\subset I$ and $\operatorname{supp}\mu\subset\mathbb{R}\setminus I$:

$$\sup_{y,z\in J}\frac{H\mu(y)-H\mu(z)}{y-z}=\int_{\mathbb{R}\setminus I}\left\{\frac{\frac{1}{x-y}-\frac{1}{x-z}}{y-z}\right\}d\mu(x)=\int_{\mathbb{R}\setminus I}\frac{1}{(x-y)(x-z)}d\mu(x)\approx\frac{\mathrm{P}(J,\mu)}{|J|}.\tag{3.4}$$

We wish to obtain a similar control in the situation at hand, but the matter is now complicated by the extra dimensions. Fix $y=\left(y^{1},y'\right),z=\left(z^{1},z'\right)\in J$ and $x=\left(x^{1},0\right)\in L\setminus\gamma J$.

We consider first the case

$$\left|y'-z'\right|\leq\left|y^{1}-z^{1}\right|,\tag{3.5}$$

We pause to recall the main assumption in (3.2) regarding the size of the graphing function:

$$\left\|D\psi\right\|_{\infty}<\frac{1}{8n^{2}}(n-\alpha).\tag{3.6}$$

Now the first component $\left(\mathbf{R}_{\Psi}^{\alpha,n}\right)_{1}$ is 'positive' in the direction of the x^{1}-axis L, and so for $\left(y^{1},y'\right),\left(z^{1},z'\right)\in J$, we write

$$\begin{aligned}
&\frac{\left(\mathbf{R}_{\Psi}^{\alpha,n}\right)_{1}\left(\mathbf{1}_{I\setminus\gamma J}\omega\right)\left(y^{1},y'\right)-\left(\mathbf{R}_{\Psi}^{\alpha,n}\right)_{1}\left(\mathbf{1}_{I\setminus\gamma J}\omega\right)\left(z^{1},z'\right)}{y^{1}-z^{1}}\\
=\;&\int_{I\setminus\gamma J}\left\{\frac{\left(\mathbf{K}_{\Psi}^{\alpha,n}\right)_{1}\left(\left(y^{1},y'\right),x\right)-\left(\mathbf{K}_{\Psi}^{\alpha,n}\right)_{1}\left(\left(z^{1},z'\right),x\right)}{y^{1}-z^{1}}\right\}d\omega(x)\\
=\;&\int_{I\setminus\gamma J}\left\{\frac{\frac{y^{1}-x^{1}}{|\Psi(y)-\Psi(x)|^{n+1-\alpha}}-\frac{z^{1}-x^{1}}{|\Psi(z)-\Psi(x)|^{n+1-\alpha}}}{y^{1}-z^{1}}\right\}d\omega(x)\ .
\end{aligned}$$

For $0\leq t\leq1$ define

$$\begin{aligned}
w_{t}&\equiv ty+(1-t)z=z+t(y-z),\\
\text{so that }w_{t}-x&=t(y-x)+(1-t)(z-x),
\end{aligned}$$

and

$$\Phi(t)\equiv\frac{w_{t}^{1}-x^{1}}{\left|\Psi(w_{t})-\Psi(x)\right|^{n+1-\alpha}},$$

so that

$$\frac{y^1-x^1}{\left|\Psi(y)-\Psi(x)\right|^{n+1-\alpha}}-\frac{z^1-x^1}{\left|\Psi(z)-\Psi(x)\right|^{n+1-\alpha}}=\Phi(1)-\Phi(0)=\int_0^1\Phi'(t)\,dt\ .$$

Now we will use (3.1) and $\nabla\left|\xi\right|^{\tau}=\tau\left|\xi\right|^{\tau-2}\xi$ to compute that

$$\begin{aligned}\frac{d}{dt}\Phi(t) &= \frac{y^1-z^1}{\left|\Psi(w_t)-\Psi(x)\right|^{n+1-\alpha}}\\ &\quad+\left(w_t^1-x^1\right)\frac{-(n+1-\alpha)}{\left|\Psi(w_t)-\Psi(x)\right|^{n+3-\alpha}}\\ &\quad\times\left(\begin{array}{cc} w_t^1-x^1, & w_t'-x'+\psi\left(w_t^1\right)-\psi\left(x^1\right)\end{array}\right)\left[\begin{array}{cc}1 & \mathbf{0}\\ D\psi\left(w_t^1\right) & \mathbf{I}_{n-1}\end{array}\right]\left(\begin{array}{c}y^1-z^1\\ y'-z'\end{array}\right),\end{aligned}$$

where $\mathbf{I}_{n-1}$ denotes the $(n-1)\times(n-1)$ identity matrix. Thus we have

$$\begin{aligned}\frac{d}{dt}\Phi(t) &= \frac{y^1-z^1}{\left|\Psi(w_t)-\Psi(x)\right|^{n+1-\alpha}}-(n+1-\alpha)\left(w_t^1-x^1\right)\frac{\left(w_t^1-x^1\right)\left(y^1-z^1\right)}{\left|\Psi(w_t)-\Psi(x)\right|^{n+3-\alpha}}\\ &\quad-(n+1-\alpha)\left(w_t^1-x^1\right)\frac{\left(w_t'-x'+\psi\left(w_t^1\right)-\psi\left(x^1\right)\right)\cdot\left(y'-z'\right)}{\left|\Psi(w_t)-\Psi(x)\right|^{n+3-\alpha}}\\ &\quad-(n+1-\alpha)\left(w_t^1-x^1\right)\frac{\left(w_t'-x'+\psi\left(w_t^1\right)-\psi\left(x^1\right)\right)\cdot D\psi\left(w_t^1\right)\left(y^1-z^1\right)}{\left|\Psi(w_t)-\Psi(x)\right|^{n+3-\alpha}}\\ &= \left(y^1-z^1\right)\left\{\frac{\left|\Psi(w_t)-\Psi(x)\right|^2}{\left|\Psi(w_t)-\Psi(x)\right|^{n+3-\alpha}}-(n+1-\alpha)\frac{\left|w_t^1-x^1\right|^2}{\left|\Psi(w_t)-\Psi(x)\right|^{n+3-\alpha}}\right\}\\ &\quad+\left(y^1-z^1\right)\left\{-(n+1-\alpha)\left(w_t^1-x^1\right)\frac{\left(w_t'-x'+\psi\left(w_t^1\right)-\psi\left(x^1\right)\right)\cdot\left(\frac{y'-z'}{y^1-z^1}\right)}{\left|\Psi(w_t)-\Psi(x)\right|^{n+3-\alpha}}\right\}\\ &\quad+\left(y^1-z^1\right)\left\{-(n+1-\alpha)\left(w_t^1-x^1\right)\frac{\left(w_t'-x'+\psi\left(w_t^1\right)-\psi\left(x^1\right)\right)\cdot D\psi\left(w_t^1\right)}{\left|\Psi(w_t)-\Psi(x)\right|^{n+3-\alpha}}\right\}\\ &\equiv \left(y^1-z^1\right)\left\{A(t)+B(t)+C(t)\right\}.\end{aligned}$$

From (3.6) we have $\left\|D\psi\right\|_\infty<\frac{1}{8n^2}(n-\alpha)$. Now $\left|w_t^1-x^1\right|\approx\left|y-x\right|$ and $\left|w_t'-x'\right|=\left|w_t'\right|\lesssim\frac{|y-x|}{\gamma}$ because $x\in L\setminus\gamma J$ and $y,z\in J$ and $3J\cap L\neq\emptyset$, and so we obtain from (3.2), with $\gamma=\gamma(n,\alpha)$ sufficiently large, that both

$$\begin{aligned}\left|w_t'-x'\right| &\leq \frac{1}{4}\sqrt{n-\alpha}\left|w_t^1-x^1\right|,\\ \left|\psi\left(w_t^1\right)-\psi\left(x^1\right)\right| &\leq \left\|D\psi\right\|_\infty\left|w_t^1-x^1\right|\leq\frac{1}{4}\sqrt{n-\alpha}\left|w_t^1-x^1\right|.\end{aligned}\tag{3.7}$$

Hence we have

$$\begin{aligned}
-A(t) &= -\frac{|\Psi(w_t)-\Psi(x)|^2}{|\Psi(w_t)-\Psi(x)|^{n+3-\alpha}} + (n+1-\alpha)\frac{\left|w_t^1-x^1\right|^2}{|\Psi(w_t)-\Psi(x)|^{n+3-\alpha}} \\
&= \frac{-|\Psi(w_t)-\Psi(x)|^2 + (n+1-\alpha)\left|w_t^1-x^1\right|^2}{|\Psi(w_t)-\Psi(x)|^{n+3-\alpha}} \\
&= \frac{-\left|w_t'-x'+\psi\left(w_t^1\right)-\psi\left(x^1\right)\right|^2 + (n-\alpha)\left(w_t^1-x^1\right)^2}{|\Psi(w_t)-\Psi(x)|^{n+3-\alpha}} \\
&\geq \frac{3}{4}(n-\alpha)\frac{\left(w_t^1-x^1\right)^2}{|\Psi(w_t)-\Psi(x)|^{n+3-\alpha}} \ ,
\end{aligned}$$

where the inequality in the final line holds because of (3.7). Note that we are able to control the sign of $A(t)$ above by using the hypothesis that $\|D\psi\|_\infty$ is small to keep $\left|\psi\left(w_t^1\right)-\psi\left(x^1\right)\right|$ sufficiently small, and then using the hypothesis that γ is large to keep $|w_t'-x'|$ sufficiently small, so that altogether $(n-\alpha)\left(w_t^1-x^1\right)^2$ is the dominant term in the numerator.

Now from our assumption (3.5) and (3.6), i.e. $\|D\psi\|_\infty < \frac{1}{8n^2}(n-\alpha)$, we have

$$\begin{aligned}
|B(t)| &= \left|(n+1-\alpha)\left(w_t^1-x^1\right)\frac{\left(w_t'-x'+\psi\left(w_t^1\right)-\psi\left(x^1\right)\right)\cdot\left(\frac{y'-z'}{y^1-z^1}\right)}{|\Psi(w_t)-\Psi(x)|^{n+3-\alpha}}\right| \\
&\leq (n+1-\alpha)\left|w_t^1-x^1\right|\frac{\left(|w_t'-x'|+\|D\psi\|_\infty\left|w_t^1-x^1\right|\right)\frac{|y'-z'|}{|y^1-z^1|}}{|\Psi(w_t)-\Psi(x)|^{n+3-\alpha}} \\
&\leq (n+1-\alpha)\frac{\left|w_t^1-x^1\right|\left(|w_t'-x'|+\|D\psi\|_\infty\left|w_t^1-x^1\right|\right)}{|\Psi(w_t)-\Psi(x)|^{n+3-\alpha}} \\
&\leq (n+1-\alpha)\left(\frac{a_{n,\alpha}}{\gamma}+\frac{1}{8n^2}(n-\alpha)\right)\frac{\left|w_t^1-x^1\right|^2}{|\Psi(w_t)-\Psi(x)|^{n+3-\alpha}} \\
&\leq \frac{1}{4}(n-\alpha)\frac{\left(w_t^1-x^1\right)^2}{|\Psi(w_t)-\Psi(x)|^{n+3-\alpha}},
\end{aligned}$$

with a constant $a_{n,\alpha}$ independent of γ, provided

$$\frac{a_{n,\alpha}}{\gamma}+\frac{1}{8n^2}(n-\alpha) \leq \frac{1}{4}\frac{n-\alpha}{n+1-\alpha}, \tag{3.8}$$

which holds for $\gamma = \gamma(n,\alpha)$ sufficiently large. We also have from the same calculation that

$$\begin{aligned}
|C(t)| &= \left|(n+1-\alpha)\left(w_t^1 - x^1\right)\frac{\left(w_t' - x' + \psi\left(w_t^1\right) - \psi\left(x^1\right)\right)\cdot D\psi\left(w_t^1\right)}{\left|\Psi(w_t) - \Psi(x)\right|^{n+3-\alpha}}\right| \\
&\leq (n+1-\alpha)\left|w_t^1 - x^1\right|\frac{\left(\left|w_t' - x'\right| + \|D\psi\|_\infty\left|w_t^1 - x^1\right|\right)\|D\psi\|_\infty}{\left|\Psi(w_t) - \Psi(x)\right|^{n+3-\alpha}} \\
&\leq \frac{1}{4}(n-\alpha)\frac{\left(w_t^1 - x^1\right)^2}{\left|\Psi(w_t) - \Psi(x)\right|^{n+3-\alpha}}\|D\psi\|_\infty \\
&\leq \frac{1}{4}(n-\alpha)\frac{\left(w_t^1 - x^1\right)^2}{\left|\Psi(w_t) - \Psi(x)\right|^{n+3-\alpha}}\frac{1}{8n^2}(n-\alpha) \\
&< \frac{1}{4}(n-\alpha)\frac{\left(w_t^1 - x^1\right)^2}{\left|\Psi(w_t) - \Psi(x)\right|^{n+3-\alpha}}.
\end{aligned}$$

Thus altogether in case (3.5) we have

$$\begin{aligned}
&\left|\left(\mathbf{R}_\Psi^{\alpha,n}\right)_1\left(\mathbf{1}_{I\setminus\gamma J}\omega\right)\left(y^1,y'\right) - \left(\mathbf{R}_\Psi^{\alpha,n}\right)_1\left(\mathbf{1}_{I\setminus\gamma J}\omega\right)\left(z^1,z'\right)\right| \\
= \;& \left|y^1 - z^1\right|\left|\int_{I\setminus\gamma J}\frac{\int_0^1 \frac{d}{dt}\Phi(t)\,dt}{y^1 - z^1}d\omega(x)\right| \\
= \;& \left|y^1 - z^1\right|\left|\int_{I\setminus\gamma J}\int_0^1\left\{A(t) + B(t) + C(t)\right\}dtd\omega(x)\right| \\
\gtrsim \;& \left|y^1 - z^1\right|\left|\int_{I\setminus\gamma J}\int_0^1\left\{(n-\alpha)\frac{\left(w_t^1 - x^1\right)^2}{\left|\Psi(w_t) - \Psi(x)\right|^{n+3-\alpha}}\right\}dtd\omega(x)\right| \\
\approx \;& \left|y^1 - z^1\right|(n-\alpha)\int_{I\setminus\gamma J}\frac{\left(c_J^1 - x^1\right)^2}{\left|c_J - x\right|^{n+3-\alpha}}d\omega(x) \\
\approx \;& \left|y^1 - z^1\right|\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus\gamma J}\omega\right)}{|J|^{\frac{1}{n}}},
\end{aligned}$$

where the constants implicit in $\approx$ depend only on n and α.

On the other hand, in the case that

$$\left|y' - z'\right| > \left|y^1 - z^1\right|, \tag{3.9}$$

we write

$$\begin{aligned}
\left(\mathbf{R}^{\alpha,n}\right)' &= \left(R_2^{\alpha,n},...,R_n^{\alpha,n}\right), \\
\mathbf{\Phi}(t) &= \frac{w_t' - x'}{\left|\Psi(w_t) - \Psi(x)\right|^{n+1-\alpha}},
\end{aligned}$$

with $w_t = ty + (1-t)z$ as before. Then as above we obtain

$$\frac{y' - x'}{\left|\Psi(y) - \Psi(x)\right|^{n+1-\alpha}} - \frac{z' - x'}{\left|\Psi(z) - \Psi(x)\right|^{n+1-\alpha}} = \mathbf{\Phi}(1) - \mathbf{\Phi}(0) = \int_0^1 \frac{d}{dt}\mathbf{\Phi}(t)\,dt,$$

where if we write $\widehat{y^k} \equiv \left(0, y^2, ..., y^{k-1}, 0, y^{k+1}, ..., y^n\right)$, we have, similarly to the computation of $\frac{d}{dt}\Phi(t)$ above,

$$\begin{aligned}
&\frac{d}{dt}\mathbf{\Phi}(t) \equiv \left\{\frac{d}{dt}\Phi_k(t)\right\}_{k=2}^n \\
= &\left\{\left(y^k - z^k\right)\left[\frac{\left|\Psi(w_t) - \Psi(x)\right|^2}{\left|\Psi(w_t) - \Psi(x)\right|^{n+3-\alpha}} - (n+1-\alpha)\frac{\left(w_t^k - x^k\right)\left[w_t^k - x^k + \psi^k\left(w_t^1\right) - \psi^k\left(x^1\right)\right]}{\left|\Psi(w_t) - \Psi(x)\right|^{n+3-\alpha}}\right]\right\}_{k=2}^n \\
&- \left\{(n+1-\alpha)\left(w_t^k - x^k\right)\frac{\left(w_t^1 - x^1\right)\left(y^1 - z^1\right) + \left(\widehat{w_t^k} - \widehat{x^k} + \widehat{\psi^k}\left(w_t^1\right) - \widehat{\psi^k}\left(x^1\right)\right)\cdot\left(\widehat{y^k} - \widehat{z^k}\right)}{\left|\Psi(w_t) - \Psi(x)\right|^{n+3-\alpha}}\right\}_{k=2}^n \\
&- \left\{\left(y^1 - z^1\right)\left[-(n+1-\alpha)\left(w_t^k - x^k\right)\frac{\left(w_t' - x' + \psi\left(w_t^1\right) - \psi\left(x^1\right)\right)\cdot D\psi\left(w_t^1\right)}{\left|\Psi(w_t) - \Psi(x)\right|^{n+3-\alpha}}\right]\right\}_{k=2}^n \\
\equiv &\left\{\left(y^k - z^k\right)A_k(t)\right\}_{k=2}^n + \left\{V_k(t)\right\}_{k=2}^n + \left\{\left(y^1 - z^1\right)C_k(t)\right\}_{k=2}^n \equiv \mathbf{U}(t) + \mathbf{V}(t) + \mathbf{W}(t)\ .
\end{aligned}$$

Now for $2 \leq k \leq n$ we have $x^k = 0$ and so

$$\begin{aligned}
A_k(t) &= \frac{\left|\Psi(w_t) - \Psi(x)\right|^2}{\left|\Psi(w_t) - \Psi(x)\right|^{n+3-\alpha}} - (n+1-\alpha)\frac{w_t^k\left[w_t^k + \psi^k\left(w_t^1\right) - \psi^k\left(x^1\right)\right]}{\left|\Psi(w_t) - \Psi(x)\right|^{n+3-\alpha}} \\
&= \frac{\left|\Psi(w_t) - \Psi(x)\right|^2 - (n+1-\alpha)w_t^k\left[w_t^k + \psi^k\left(w_t^1\right) - \psi^k\left(x^1\right)\right]}{\left|\Psi(w_t) - \Psi(x)\right|^{n+3-\alpha}} \\
&= \frac{\left|w_t^1 - x^1\right|^2 + \sum_{j\neq 1}\left|w_t^j + \psi^j\left(w_t^1\right) - \psi^j\left(x^1\right)\right|^2 - (n+1-\alpha)w_t^k\left(w_t^k + \psi^k\left(w_t^1\right) - \psi^k\left(x^1\right)\right)}{\left|\Psi(w_t) - \Psi(x)\right|^{n+3-\alpha}}.
\end{aligned} \tag{3.10}$$

Then using $\left|w_t^k\right| \lesssim \frac{1}{\gamma}\left|w_t^1 - x^1\right|$ and $\left|\psi^k\left(w_t^1\right) - \psi^k\left(x^1\right)\right| \leq \|D\psi\|_\infty\left|w_t^1 - x^1\right|$, we claim

$$A_k(t) \geq \frac{1}{2}\frac{\left|w_t^1 - x^1\right|^2}{\left|\Psi(w_t) - \Psi(x)\right|^{n+3-\alpha}},$$

where $\|D\psi\|_\infty$ satisfies (3.2) and $\gamma = \gamma(n,\alpha)$ is sufficiently large. Indeed, use $\left|w_t^k\right| \leq \frac{b_{n,\alpha}}{\gamma}\left|w_t^1 - x^1\right|$, where the constant $b_{n,\alpha}$ is independent of γ, to obtain

$$\begin{aligned}
&\left|\Psi(w_t) - \Psi(x)\right|^{n+3-\alpha}A_k(t) \\
\geq\ &\left|w_t^1 - x^1\right|^2 - (n+1-\alpha)\left|w_t^k\right|\left|w_t^k + \psi^k\left(w_t^1\right) - \psi^k\left(x^1\right)\right| \\
\geq\ &\left|w_t^1 - x^1\right|^2 - (n+1-\alpha)\left[\left|w_t^k\right|^2 + \left|w_t^k\right|\|D\psi\|_\infty\left|w_t^1 - x^1\right|\right] \\
\geq\ &\left|w_t^1 - x^1\right|^2 - (n+1-\alpha)\left[\left(\frac{b_{n,\alpha}}{\gamma}\right)^2 + \frac{b_{n,\alpha}}{\gamma}\|D\psi\|_\infty\right]\left|w_t^1 - x^1\right|^2 \\
\geq\ &\frac{1}{2}\left|w_t^1 - x^1\right|^2,
\end{aligned}$$

for $\gamma = \gamma(n,\alpha)$ sufficiently large since $\|D\psi\|_\infty < \frac{n-\alpha}{8n^2}$ by (3.2).

Thus we have

$$\int_{I\setminus\gamma J} A_k(t)\, d\omega(x) \geq \frac{1}{2} c_{n,\alpha} \int_{I\setminus\gamma J} \frac{1}{|c_J - x|^{n+1-\alpha}} d\omega(x) \geq c'_{n,\alpha} \frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_{I\setminus\gamma J}\omega\right)}{|J|^{\frac{1}{n}}},$$

where $c'_{n,\alpha}$ is *independent* of the choice of $\gamma = \gamma(n,\alpha)$, and hence

$$\begin{aligned}
\left| \int_{I\setminus\gamma J} \int_0^1 \mathbf{U}(t)\, dt d\omega(x) \right|^2 &= \left| \int_{I\setminus\gamma J} \int_0^1 \left\{ \left(y^k - z^k\right) A_k(t) \right\}_{k=2}^n dt d\omega(x) \right|^2 \\
&= \sum_{k=2}^n \left(y^k - z^k\right)^2 \left| \int_{I\setminus\gamma J} \int_0^1 A_k(t)\, dt d\omega(x) \right|^2 \\
&\geq \left(c'_{n,\alpha}\right)^2 \sum_{k=2}^n \left(y^k - z^k\right)^2 \left(\frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_{I\setminus\gamma J}\omega\right)}{|J|^{\frac{1}{n}}} \right)^2 \\
&= \left(c'_{n,\alpha}\right)^2 |y' - z'|^2 \left(\frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_{I\setminus\gamma J}\omega\right)}{|J|^{\frac{1}{n}}} \right)^2 .
\end{aligned}$$

For $2 \leq k \leq n$ we also have using $x^k = 0$ and (3.9) that

$$\begin{aligned}
&\frac{1}{n+1-\alpha} |V_k(t)| \\
= &\left| \left(w_t^k - x^k\right) \frac{\left(w_t^1 - x^1\right)\left(y^1 - z^1\right) + \left(\widehat{w_t^k} - \widehat{x^k} + \widehat{\psi^k}\left(w_t^1\right) - \widehat{\psi^k}\left(x^1\right)\right) \cdot \left(\widehat{y^k} - \widehat{z^k}\right)}{|\Psi(w_t) - \Psi(x)|^{n+3-\alpha}} \right| \\
\leq &\left|w_t^k\right| \frac{\left|w_t^1 - x^1\right| \left|y^1 - z^1\right| + \sum_{j\neq 1,k} \left| w_t^j + \psi^j\left(w_t^1\right) - \psi^j\left(x^1\right) \right| \left|y^j - z^j\right|}{|\Psi(w_t) - \Psi(x)|^{n+3-\alpha}} \\
\leq &\left\{ \frac{\left|w_t^k\right| \left|y^1 - z^1\right|}{|\Psi(w_t) - \Psi(x)|^{n+2-\alpha}} + \sum_{j\neq 1,k} \frac{\left|w_t^k\right| \left(\left|w_t^j\right| + \|D\psi\|_\infty \left|w_t^1 - x^1\right| \right) \left|y^j - z^j\right|}{|\Psi(w_t) - \Psi(x)|^{n+3-\alpha}} \right\} \\
\lesssim &\left\{ \frac{b_{n,\alpha}}{\gamma} \frac{\left|y^1 - z^1\right|}{|\Psi(w_t) - \Psi(x)|^{n+1-\alpha}} + \left(\sqrt{n}\ell(J)\right) \left(\sqrt{n}\ell(J) + \|D\psi\|_\infty \left|w_t^1 - x^1\right|\right) \frac{|y' - z'|}{|\Psi(w_t) - \Psi(x)|^{n+3-\alpha}} \right\} \\
\lesssim &\left\{ \frac{b_{n,\alpha}}{\gamma} \frac{|y' - z'|}{|c_J - x|^{n+1-\alpha}} + \left(\frac{b_{n,\alpha}}{\gamma}\right) \left(\frac{b_{n,\alpha}}{\gamma} + \|D\psi\|_\infty\right) \frac{|y' - z'|}{|c_J - x|^{n+1-\alpha}} \right\},
\end{aligned}$$

as well as

$$\begin{aligned}
& \frac{1}{n+1-\alpha}\left|W_k\left(t\right)\right| = \frac{1}{n+1-\alpha}\left|y^1-z^1\right|\left|C_k\left(t\right)\right| \\
= & \left|y^1-z^1\right|\left|w_t^k-x^k\right|\left|\frac{\left(w_t'-x'+\psi\left(w_t^1\right)-\psi\left(x^1\right)\right)\cdot D\psi\left(w_t^1\right)}{\left|\Psi\left(w_t\right)-\Psi\left(x\right)\right|^{n+3-\alpha}}\right| \\
\leq & \left|y^1-z^1\right|\left|w_t^k\right|\frac{\left(\left|w_t'\right|+\left|\psi\left(w_t^1\right)-\psi\left(x^1\right)\right|\right)\left|D\psi\left(w_t^1\right)\right|}{\left|\Psi\left(w_t\right)-\Psi\left(x\right)\right|^{n+3-\alpha}} \\
\leq & \left|y^1-z^1\right|\left(\frac{b_{n,\alpha}^2}{\gamma^2}+\frac{b_{n,\alpha}\left\|D\psi\right\|_\infty}{\gamma}\right)\left\|D\psi\right\|_\infty\frac{\left|w_t^1-x^1\right|^2}{\left|\Psi\left(w_t\right)-\Psi\left(x\right)\right|^{n+3-\alpha}} \\
\lesssim & \left|y^1-z^1\right|\left(\frac{1}{\gamma^2}+\frac{\left\|D\psi\right\|_\infty}{\gamma}\right)\left\|D\psi\right\|_\infty\frac{\left|w_t^1-x^1\right|}{\left|c_J-x\right|^{n+1-\alpha}} \lesssim \frac{1}{\gamma}\frac{\left|y'-z'\right|}{\left|c_J-x\right|^{n+1-\alpha}}.
\end{aligned}$$

Thus

$$\begin{aligned}
& \left|\int_{I\setminus\gamma J}\int_0^1 \mathbf{V}\left(t\right)dtd\omega\left(x\right)\right| \\
\leq & d_{n,\alpha}\left(n+1-\alpha\right)\left\{\frac{1}{\gamma}+\left(\frac{1}{\gamma}\right)\left(\frac{1}{\gamma}+\left\|D\psi\right\|_\infty\right)\right\}\int_{I\setminus\gamma J}\frac{\left|y'-z'\right|}{\left|c_J-x\right|^{n+1-\alpha}}d\omega\left(x\right) \\
\leq & d'_{n,\alpha}\left(n+1-\alpha\right)\left\{\frac{1}{\gamma}+\left(\frac{1}{\gamma}\right)\left(\frac{1}{\gamma}+\frac{n-\alpha}{8n^2}\right)\right\}\left|y'-z'\right|\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus\gamma J}\omega\right)}{\left|J\right|^{\frac{1}{n}}},
\end{aligned}$$

and

$$\left|\int_{I\setminus\gamma J}\int_0^1 \mathbf{W}\left(t\right)dtd\omega\left(x\right)\right| \leq \frac{d''_{n,\alpha}}{\gamma}\frac{\left|y'-z'\right|}{\left|c_J-x\right|^{n+1-\alpha}},$$

where the constants $d'_{n,\alpha}$ and $d''_{n,\alpha}$ are independent of γ, and so we conclude that

$$\left|\int_{I\setminus\gamma J}\int_0^1 \mathbf{V}\left(t\right)dtd\omega\left(x\right)\right|+\left|\int_{I\setminus\gamma J}\int_0^1 \mathbf{W}\left(t\right)dtd\omega\left(x\right)\right| \leq \frac{1}{2}\left|\int_{I\setminus\gamma J}\int_0^1 \mathbf{U}\left(t\right)dtd\omega\left(x\right)\right|,$$

provided $\gamma = \gamma(n,\alpha)$ is sufficiently large. Then if (3.9) holds with γ sufficiently large, we have

$$\begin{aligned}
&(3.11)\\
&\left|\left(\mathbf{R}_\Psi^{\alpha,n}\right)'\mathbf{1}_{I\setminus\gamma J}\omega\left(y^1,y'\right)-\left(\mathbf{R}_\Psi^{\alpha,n}\right)'\mathbf{1}_{I\setminus\gamma J}\omega\left(z^1,z'\right)\right|\\
=&\left|\int_{I\setminus\gamma J}\left\{\frac{y^k-x^k}{\left|\Psi(y)-\Psi(x)\right|^{n+1-\alpha}}-\frac{z^k-x^k}{\left|\Psi(z)-\Psi(x)\right|^{n+1-\alpha}}\right\}_{k=2}^n d\omega(x)\right|\\
=&\left|\int_{I\setminus\gamma J}\int_0^1\mathbf{\Phi}'(t)\,dtd\omega(x)\right|\\
\geq&\left|\int_{I\setminus\gamma J}\int_0^1\mathbf{U}(t)\,dtd\omega(x)\right|-\left|\int_{I\setminus\gamma J}\int_0^1\mathbf{V}(t)\,dtd\omega(x)\right|-\left|\int_{I\setminus\gamma J}\int_0^1\mathbf{W}(t)\,dtd\omega(x)\right|\\
\geq&\frac{1}{2}\left|\int_{I\setminus\gamma J}\int_0^1\mathbf{U}(t)\,dtd\omega(x)\right|\\
\gtrsim&\left|y'-z'\right|\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus\gamma J}\omega\right)}{|J|^{\frac{1}{n}}}\gtrsim\left|y^1-z^1\right|\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus\gamma J}\omega\right)}{|J|^{\frac{1}{n}}}\ .
\end{aligned}$$

Combining the inequalities from each case (3.5) and (3.9) above, and assuming γ sufficiently large, we conclude that for all $y,z\in J$ we have the following 'strong reversal' of the 1-partial quasienergy,

$$\left|y^1-z^1\right|^2\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus\gamma J}\omega\right)}{|J|^{\frac{1}{n}}}\right)^2\lesssim\left|\mathbf{R}_\Psi^{\alpha,n}\mathbf{1}_{I\setminus\gamma J}\omega\left(y^1,y'\right)-\mathbf{R}_\Psi^{\alpha,n}\mathbf{1}_{I\setminus\gamma J}\omega\left(z^1,z'\right)\right|^2.$$

Thus we have

$$\begin{aligned}
&\sum_{\substack{J\in\mathcal{M}_{\mathrm{deep}}\\3J\cap L\neq\emptyset}}\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus\gamma J}\omega\right)}{|J|^{\frac{1}{n}}}\right)^2\int_J\left|y^1-\mathbb{E}_J^\sigma y^1\right|^2d\sigma(y)\\
=&\frac{1}{2}\sum_{\substack{J\in\mathcal{M}_{\mathrm{deep}}\\3J\cap L\neq\emptyset}}\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus\gamma J}\omega\right)}{|J|^{\frac{1}{n}}}\right)^2\frac{1}{|J|_\sigma}\int_J\int_J\left(y^1-z^1\right)^2d\sigma(y)\,d\sigma(z)\\
\lesssim&\sum_{\substack{J\in\mathcal{M}_{\mathrm{deep}}\\3J\cap L\neq\emptyset}}\frac{1}{|J|_\sigma}\int_J\int_J\left|\mathbf{R}_\Psi^{\alpha,n}\left(\mathbf{1}_{I\setminus\gamma J}\omega\right)\left(y^1,y'\right)-\mathbf{R}_\Psi^{\alpha,n}\left(\mathbf{1}_{I\setminus\gamma J}\omega\right)\left(z^1,z'\right)\right|^2d\sigma(y)\,d\sigma(z)\\
\lesssim&\sum_{\substack{J\in\mathcal{M}_{\mathrm{deep}}\\3J\cap L\neq\emptyset}}\int_J\left|\mathbf{R}_\Psi^{\alpha,n}\left(\mathbf{1}_{I\setminus\gamma J}\omega\right)\left(y^1,y'\right)\right|^2d\sigma(y)\\
\lesssim&\sum_{\substack{J\in\mathcal{M}_{\mathrm{deep}}\\3J\cap L\neq\emptyset}}\int_J\left|\mathbf{R}_\Psi^{\alpha,n}\left(\mathbf{1}_I\omega\right)\left(y^1,y'\right)\right|^2d\sigma(y)+\sum_{\substack{J\in\mathcal{M}_{\mathrm{deep}}\\3J\cap L\neq\emptyset}}\int_J\left|\mathbf{R}_\Psi^{\alpha,n}\left(\mathbf{1}_{\gamma J}\omega\right)\left(y^1,y'\right)\right|^2d\sigma(y)\,,
\end{aligned}$$

and now we obtain in the usual way that this is bounded by

$$\begin{aligned}&\int_I \left|\mathbf{R}_\Psi^{\alpha,n}\left(\mathbf{1}_I\omega\right)\left(y^1,y'\right)\right|^2 d\sigma\left(y\right)+\sum_{J\in\mathcal{M}}\left(\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\mathrm{dual}}\right)^2\left|\gamma J\right|_\omega\\ \leq\;&\left(\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\mathrm{dual}}\right)^2\left|I\right|_\omega+\beta\left(\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\mathrm{dual}}\right)^2\left|I\right|_\omega\lesssim\left(\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\mathrm{dual}}\right)^2\left|I\right|_\omega\ .\end{aligned}$$

Now we turn to the other partial quasienergies and begin with the estimate that for $2\leq j\leq n$, we have the following 'weak reversal' of energy,

$$\begin{aligned}\left|\left(\mathbf{R}_\Psi^{\alpha,n}\right)_j\left(\mathbf{1}_{I\setminus\gamma J}\omega\right)\left(y\right)\right|&=\left|\int_{I\setminus\gamma J}\frac{y^j-0}{\left|\Psi\left(y\right)-\Psi\left(x\right)\right|^{n+1-\alpha}}d\omega\left(x_1,0...,0\right)\right|\\ &=\left|\frac{y^j}{\left|J\right|^{\frac{1}{n}}}\int_{I\setminus\gamma J}\frac{\left|J\right|^{\frac{1}{n}}}{\left|\Psi\left(y\right)-\Psi\left(x\right)\right|^{n+1-\alpha}}d\omega\left(x_1,0...,0\right)\right|\approx\left|y^j\right|\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus\gamma J}\omega\right)}{\left|J\right|^{\frac{1}{n}}}.\end{aligned}\tag{3.12}$$

Thus for $2\leq j\leq n$, we use $\int_J\left|y^j-\mathbb{E}_J^\sigma y^j\right|^2 d\sigma\left(y\right)\leq\int_J\left|y^j\right|^2 d\sigma\left(y\right)$ to obtain in the usual way

$$\begin{aligned}&\sum_{\substack{J\in\mathcal{M}_{\mathrm{deep}}\\ 3J\cap L\neq\emptyset}}\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus\gamma J}\omega\right)}{\left|J\right|^{\frac{1}{n}}}\right)^2\int_J\left|y^j-\mathbb{E}_J^\sigma y^j\right|^2 d\sigma\left(y\right)\\ \leq\;&\sum_{\substack{J\in\mathcal{M}_{\mathrm{deep}}\\ 3J\cap L\neq\emptyset}}\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus\gamma J}\omega\right)}{\left|J\right|^{\frac{1}{n}}}\right)^2\int_J\left|y^j\right|^2 d\sigma\left(y\right)=\sum_{\substack{J\in\mathcal{M}_{\mathrm{deep}}\\ 3J\cap L\neq\emptyset}}\int_J\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus\gamma J}\omega\right)}{\left|J\right|^{\frac{1}{n}}}\right)^2\left|y^j\right|^2 d\sigma\left(y\right)\\ \lesssim\;&\sum_{\substack{J\in\mathcal{M}_{\mathrm{deep}}\\ 3J\cap L\neq\emptyset}}\int_J\left|\left(\mathbf{R}_\Psi^{\alpha,n}\right)_j\left(\mathbf{1}_{I\setminus\gamma J}\omega\right)\left(y\right)\right|^2 d\sigma\left(y\right)\lesssim\left(\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{dual}}\right)^2\left|I\right|_\omega+\sum_{J\in\mathcal{M}}\left(\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{dual}}\right)^2\left|\gamma J\right|_\omega\\ \leq\;&\left(\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{dual}}\right)^2\left|I\right|_\omega+\beta\left(\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{dual}}\right)^2\left|I\right|_\omega\lesssim\left(\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{dual}}\right)^2\left|I\right|_\omega\ .\end{aligned}\tag{3.13}$$

Summing these estimates for $j=1$ and $2\leq j\leq n$ completes the proof of the backward quasienergy condition $\mathcal{E}_\alpha^{\Omega\mathcal{Q}^n,\mathrm{dual}}\lesssim\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{dual}}$.

3.2. **Forward quasienergy condition.** Now we turn to proving the (forward) quasienergy condition $\mathcal{E}_\alpha^{\Omega\mathcal{Q}^n}\lesssim\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n}+\sqrt{\mathcal{A}_2^\alpha}$, where $\mathcal{A}_2^\alpha$ is the Muckenhoupt condition with holes. We must show

$$\sup_{\ell\geq0}\sum_{r=1}^\infty\sum_{J\in\mathcal{M}_{\mathrm{deep}}^\ell\left(I_r\right)}\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus J^*}\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}_J^\omega\mathbf{x}\right\|_{L^2(\omega)}^2\leq\left(\left(\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n}\right)^2+\mathcal{A}_2^\alpha\right)\left|I\right|_\sigma\ ,$$

for all partitions of a dyadic quasicube $I=\dot{\bigcup}_{r\geq1}I_r$ into dyadic subquasicubes I_r. We again fix $\ell\geq0$ and suppress both ℓ and $\mathbf{r}$ in the notation $\mathcal{M}_{\mathrm{deep}}\left(I_r\right)=\mathcal{M}_{\mathbf{r}-\mathrm{deep}}^\ell\left(I_r\right)$. We may assume that all the quasicubes J intersect $\operatorname{supp}\omega$, hence

that all the quasicubes I_r and J intersect L, which contains $\operatorname{supp}\omega$. We must show

$$\sum_{r=1}^{\infty}\sum_{J\in\mathcal{M}_{\mathrm{deep}}(I_r)}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{I\setminus J^*}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}_J^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\leq\left(\left(\mathfrak{T}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Omega\mathcal{Q}^n}\right)^2+\mathcal{A}_2^{\alpha}\right)|I|_{\sigma}\ .$$

Let $\mathcal{M}_{\mathrm{deep}}=\bigcup_{r=1}^{\infty}\mathcal{M}_{\mathrm{deep}}(I_r)$ as above, and with $J^*=\gamma J$ for each $J\in\mathcal{M}_{\mathrm{deep}}$, make the decomposition

$$I\setminus J^*=\mathrm{E}(J^*)\,\dot{\cup}\mathrm{S}(J^*)$$

of $I\setminus J^*$ into *end* $\mathrm{E}(J^*)$ and *side* $\mathrm{S}(J^*)$ disjoint pieces defined by

$$\begin{aligned}\mathrm{E}(J^*)&\equiv(I\setminus J^*)\cap\left\{\left(y^1,y'\right):|y'-c_J'|\leq\frac{10}{\gamma}\left|y^1-c_J^1\right|\right\};\\ \mathrm{S}(J^*)&\equiv(I\setminus J^*)\setminus\mathrm{E}(J^*)\ .\end{aligned}$$

Then it suffices to show both

$$\begin{aligned}A&\equiv\sum_{J\in\mathcal{M}_{\mathrm{deep}}}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{\mathrm{E}(J^*)}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}_J^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\leq\left(\left(\mathfrak{T}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Omega\mathcal{Q}^n}\right)^2+\mathcal{A}_2^{\alpha}\right)|I|_{\sigma}\ ,\\ B&\equiv\sum_{J\in\mathcal{M}_{\mathrm{deep}}}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{\mathrm{S}(J^*)}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}_J^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\leq\left(\left(\mathfrak{T}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Omega\mathcal{Q}^n}\right)^2+\mathcal{A}_2^{\alpha}\right)|I|_{\sigma}\ .\end{aligned}$$

Term A will be estimated in analogy with the Hilbert transform estimate (3.4), while term B will be estimated by summing Poisson tails. Both estimates rely heavily on the one-dimensional nature of the support of ω, for example $\left\|\mathsf{P}_J^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2=\left\|\mathsf{P}_J^{\omega}x^1\right\|_{L^2(\omega)}^2$. Thus in this quasienergy condition, there is only one nonvanishing partial quasienergy, namely the 1-partial quasienergy measured along the x_1-axis.

For $\left(x^1,0'\right),\left(z^1,0'\right)\in J$ in term A we first claim the following 'strong reversal' of quasienergy,

(3.14)

$$\begin{aligned}&\left|\frac{\left(\mathbf{R}_{\Psi}^{\alpha,n}\right)_1\left(\mathbf{1}_{\mathrm{E}(J^*)}\sigma\right)\left(x^1,0'\right)-\left(\mathbf{R}_{\Psi}^{\alpha,n}\right)_1\left(\mathbf{1}_{\mathrm{E}(J^*)}\sigma\right)\left(z^1,0'\right)}{x^1-z^1}\right|\\ =&\left|\int_{\mathrm{E}(J^*)}\left\{\frac{\left(\mathbf{K}_{\Psi}^{\alpha,n}\right)_1\left(\left(x^1,0'\right),y\right)-\left(\mathbf{K}_{\Psi}^{\alpha,n}\right)_1\left(\left(z^1,0'\right),y\right)}{x^1-z^1}\right\}d\sigma(y)\right|\\ =&\left|\int_{\mathrm{E}(J^*)}\left\{\frac{\frac{x^1-y^1}{\left(|x^1-y^1|^2+|\psi(x_1)-\psi(y_1)-y'|^2\right)^{\frac{n+1-\alpha}{2}}}-\frac{z^1-y^1}{\left(|z^1-y^1|^2+|\psi(z_1)-\psi(y_1)-y'|^2\right)^{\frac{n+1-\alpha}{2}}}}{x^1-z^1}\right\}d\sigma(y)\right|\\ \approx&\ \frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{\mathrm{E}(J^*)}\sigma\right)}{|J|^{\frac{1}{n}}}\ .\end{aligned}$$

Indeed, if we set $a(u) = |\psi(u+y_1) - \psi(y_1) - y'|$ and $s = x^1 - y^1$ and $t = z^1 - y^1$, then the term in braces in (3.14) is

$$\frac{\frac{x^1-y^1}{\left(|x^1-y^1|^2+|\psi(x^1)-\psi(y^1)-y'|^2\right)^{\frac{n+1-\alpha}{2}}} - \frac{z^1-y^1}{\left(|z^1-y^1|^2+|\psi(z^1)-\psi(y^1)-y'|^2\right)^{\frac{n+1-\alpha}{2}}}}{x^1-z^1}$$

$$= \frac{\frac{s}{\left(s^2+|\psi(s+y^1)-\psi(y^1)-y'|^2\right)^{\frac{n+1-\alpha}{2}}} - \frac{t}{\left(t^2+|\psi(t+y^1)-\psi(y^1)-y'|^2\right)^{\frac{n+1-\alpha}{2}}}}{s-t} = \frac{\varphi(s)-\varphi(t)}{s-t},$$

where with y fixed for the moment,

$$\begin{aligned}
\varphi(u) &= u\left(u^2+\left|\psi\left(u+y^1\right)-\psi\left(y^1\right)-y'\right|^2\right)^{-\frac{n+1-\alpha}{2}} = u\left(u^2+a(u)^2\right)^{-\frac{n+1-\alpha}{2}};\\
a(u)^2 &\equiv \left|\psi\left(u+y^1\right)-\psi\left(y^1\right)-y'\right|^2.
\end{aligned}$$

Now the derivative of $\varphi(u)$ is

$$\begin{aligned}
\frac{d}{du}\varphi(u) &= \left(u^2+a(u)^2\right)^{-\frac{n+1-\alpha}{2}} - \frac{n+1-\alpha}{2}\left(u^2+a(u)^2\right)^{-\frac{n+1-\alpha}{2}-1} 2\left[u^2+u\frac{d}{du}\frac{1}{2}a(u)^2\right]\\
&= \left(u^2+a(u)^2\right)^{-\frac{n+1-\alpha}{2}-1}\left\{\left(u^2+a(u)^2\right)-(n+1-\alpha)\left[u^2+u\frac{d}{du}\frac{1}{2}a(u)^2\right]\right\}\\
&= \left(u^2+a(u)^2\right)^{-\frac{n+1-\alpha}{2}-1}\left\{\left[a(u)^2-(n+1-\alpha)u\frac{d}{du}\frac{1}{2}a(u)^2\right]-(n-\alpha)u^2\right\},
\end{aligned}$$

and the derivative of $a(u)^2$ is

$$\begin{aligned}
\frac{d}{du}\frac{1}{2}a(u)^2 &= \frac{1}{2}\frac{d}{du}\left|\psi(u+y_1)-\psi(y_1)-y'\right|^2\\
&= \left[\psi(u+y_1)-\psi(y_1)-y'\right]\cdot D\psi(u+y_1).
\end{aligned}$$

We now want to conclude that
(3.15)

$$\left|a(u)^2-(n+1-\alpha)u\frac{d}{du}\frac{1}{2}a(u)^2\right| \le \frac{1}{2}(n-\alpha)u^2, \qquad \text{for } u \text{ between } s \text{ and } t,$$

so that $-(n-\alpha)u^2$ is the dominant term inside the braces in the formula for $\frac{d}{du}\varphi(u)$. For this we note that $|y'| \le C\frac{1}{\gamma}|u|$, and so using $2ab \le \varepsilon a^2 + \frac{1}{\varepsilon}b^2$ twice, the left side of (3.15) is at most

$$\begin{aligned}
&\left(\|D\psi\|_\infty|u|+|y'|\right)^2+(n+1-\alpha)|u|\left(\|D\psi\|_\infty|u|+|y'|\right)\|D\psi\|_\infty\\
\le\ & 2\|D\psi\|_\infty^2u^2+2|y'|^2+(n+1-\alpha)\|D\psi\|_\infty^2u^2+(n+1-\alpha)\|D\psi\|_\infty|u||y'|\\
\le\ & (n+4-\alpha)\|D\psi\|_\infty^2u^2+C_1\left(C\frac{1}{\gamma}|u|\right)^2,
\end{aligned}$$

which gives (3.15) if

$$(n+4-\alpha)\|D\psi\|_\infty^2+C_1C^2\frac{1}{\gamma^2} \le \frac{1}{2}(n-\alpha),$$

which in turn holds for $\|D\psi\|_\infty$ and $\gamma = \gamma(n,\alpha)$ satisfying

$$\|D\psi\|_\infty < \frac{1}{2}\sqrt{\frac{n-\alpha}{n+4-\alpha}} \text{ and } \gamma \gg \frac{1}{\sqrt{n-\alpha}}, \tag{3.16}$$

where the first inequality in (3.16) follows from (3.2), i.e. $\|D\psi\|_\infty < \frac{1}{8n^2}(n-\alpha) < \frac{1}{2}\sqrt{\frac{n-\alpha}{n+4-\alpha}}$, and the second inequality holds for $\gamma = \gamma(n,\alpha)$ sufficiently large.

Thus we get

$$-\frac{d}{du}\varphi(u) \approx t^2\left(t^2 + a(t)^2\right)^{-\frac{n+1-\alpha}{2}-1}, \qquad \text{for } u \text{ between } s \text{ and } t,$$

where for $(x^1, 0'), (z^1, 0') \in J$ with $J \in \mathcal{M}_{\mathrm{deep}}$, the implied constants of comparability are independent of $y \in \mathrm{E}(J^*)$. Finally, since $|s-t| \lesssim \frac{1}{\gamma}|t| \ll |t|$, the derivative $\frac{d\varphi}{du}$ is essentially constant on the small interval (s,t), and we can apply the tangent line approximation to φ to obtain $\varphi(s) - \varphi(t) \approx \frac{d\varphi}{dt}(t)(s-t)$, and conclude that for $(x^1, 0'), (z^1, 0') \in J$,

$$\begin{aligned}
&\left|\int_{\mathrm{E}(J^*)} \left\{ \frac{\frac{x^1-y^1}{\left(|x^1-y^1|^2+|\psi(x^1)-\psi(y^1)-y'|^2\right)^{\frac{n+1-\alpha}{2}}} - \frac{z^1-y^1}{\left(|z^1-y^1|^2+|\psi(z^1)-\psi(y^1)-y'|^2\right)^{\frac{n+1-\alpha}{2}}}}{x^1-z^1} \right\} d\sigma(y)\right| \\
\approx\ & \int_{\mathrm{E}(J^*)} \frac{\left|z^1-y^1\right|^2}{\left(|z^1-y^1|^2 + |\psi(z^1)-\psi(y^1)-y'|^2\right)^{\frac{n+1-\alpha}{2}+1}} d\sigma(y) \\
\approx\ & \int_{\mathrm{E}(J^*)} \frac{\left|z^1-y^1\right|^2}{\left(|z^1-y^1|^2 + |y'|^2\right)^{\frac{n+1-\alpha}{2}+1}} d\sigma(y) \approx \frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_{\mathrm{E}(J^*)}\sigma\right)}{|J|^{\frac{1}{n}}},
\end{aligned}$$

which proves (3.14).

Thus we have

$$\begin{aligned}
&(3.17) \\
&\sum_{J\in\mathcal{M}_{\mathrm{deep}}} \left(\frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_{\mathrm{E}(J^*)}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \int_{J\cap L} \left|x^1 - \mathbb{E}_J^\omega x^1\right|^2 d\omega(y) \\
=\ & \frac{1}{2}\sum_{J\in\mathcal{M}_{\mathrm{deep}}} \left(\frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_{\mathrm{E}(J^*)}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \frac{1}{|J\cap L|_\omega} \int_{J\cap L}\int_{J\cap L} \left(x^1 - z^1\right)^2 d\omega(x)\, d\omega(z) \\
\approx\ & \sum_{J\in\mathcal{M}_{\mathrm{deep}}} \frac{1}{|J|_\omega} \int_{J\cap L}\int_{J\cap L} \left\{ \left(\mathbf{R}_\Psi^{\alpha,n}\right)_1 \left(\mathbf{1}_{\mathrm{E}(J^*)}\sigma\right)\left(x^1, 0'\right) - \left(\mathbf{R}_\Psi^{\alpha,n}\right)_1 \left(\mathbf{1}_{\mathrm{E}(J^*)}\sigma\right)\left(z^1, 0'\right) \right\}^2 d\omega(x)\, d\omega(z) \\
\lesssim\ & \sum_{J\in\mathcal{M}_{\mathrm{deep}}} \frac{1}{|J|_\omega} \int_{J\cap L}\int_{J\cap L} \left\{ \left(\mathbf{R}_\Psi^{\alpha,n}\right)_1 \left(\mathbf{1}_I\sigma\right)\left(x^1, 0'\right) - \left(\mathbf{R}_\Psi^{\alpha,n}\right)_1 \left(\mathbf{1}_I\sigma\right)\left(z^1, 0'\right) \right\}^2 d\omega(x)\, d\omega(z) \\
& + \sum_{J\in\mathcal{M}_{\mathrm{deep}}} \frac{1}{|J|_\omega} \int_{J\cap L}\int_{J\cap L} \left\{ \left(\mathbf{R}_\Psi^{\alpha,n}\right)_1 \left(\mathbf{1}_{J^*}\sigma\right)\left(x^1, 0'\right) - \left(\mathbf{R}_\Psi^{\alpha,n}\right)_1 \left(\mathbf{1}_{J^*}\sigma\right)\left(z^1, 0'\right) \right\}^2 d\omega(x)\, d\omega(z) \\
& + \sum_{J\in\mathcal{M}_{\mathrm{deep}}} \frac{1}{|J|_\omega} \int_{J\cap L}\int_{J\cap L} \left\{ \left(\mathbf{R}_\Psi^{\alpha,n}\right)_1 \left(\mathbf{1}_{\mathrm{S}(J^*)}\right)\sigma\left(x^1, 0'\right) - \left(\mathbf{R}_\Psi^{\alpha,n}\right)_1 \left(\mathbf{1}_{\mathrm{S}(J^*)}\sigma\right)\left(z^1, 0'\right) \right\}^2 d\omega(x)\, d\omega(z) \\
\equiv\ & A_1 + A_2 + A_3,
\end{aligned}$$

since $I = J^* \dot{\cup} (I \setminus J^*) = J^* \dot{\cup} \mathrm{E}(J^*) \dot{\cup} \mathrm{S}(J^*)$ where $\dot{\cup}$ denotes disjoint union. Now we can discard the difference in term A_1 by writing

$$\left|(\mathbf{R}_\Psi^{\alpha,n})_1 (\mathbf{1}_I \sigma) (x^1, 0') - (\mathbf{R}_\Psi^{\alpha,n})_1 (\mathbf{1}_I \sigma) (z^1, 0')\right| \leq \left|(\mathbf{R}_\Psi^{\alpha,n})_1 (\mathbf{1}_I \sigma) (x^1, 0')\right| + \left|(\mathbf{R}_\Psi^{\alpha,n})_1 (\mathbf{1}_I \sigma) (z^1, 0')\right|$$

to obtain from pairwise disjointedness of $J \in \mathcal{M}_{\mathrm{deep}}$,

(3.18)

$$A_1 \lesssim \sum_{J \in \mathcal{M}_{\mathrm{deep}}} \int_{J \cap L} \left|(\mathbf{R}_\Psi^{\alpha,n})_1 (\mathbf{1}_I \sigma) (x^1, 0')\right|^2 d\omega(x) \leq \int_I \left|(\mathbf{R}_\Psi^{\alpha,n})_1 (\mathbf{1}_I \sigma)\right|^2 d\omega \leq \left(\mathfrak{T}^{\Omega \mathcal{Q}^n}_{(\mathbf{R}_\Psi^{\alpha,n})_1}\right)^2 |I|_\sigma ,$$

and similarly we can discard the difference in term A_2, and use the bounded overlap property (2.4), to obtain

(3.19)

$$\begin{aligned} A_2 &\lesssim \sum_{J \in \mathcal{M}_{\mathrm{deep}}} \int_{J \cap L} \left|(\mathbf{R}_\Psi^{\alpha,n})_1 (\mathbf{1}_{J^*} \sigma) (x^1, 0')\right|^2 d\omega(x) \leq \sum_{J \in \mathcal{M}_{\mathrm{deep}}} \left(\mathfrak{T}^{\Omega \mathcal{Q}^n}_{(\mathbf{R}_\Psi^{\alpha,n})_1}\right)^2 |J^*|_\sigma \\ &= \left(\mathfrak{T}^{\Omega \mathcal{Q}^n}_{(\mathbf{R}_\Psi^{\alpha,n})_1}\right)^2 \sum_{r=1}^\infty \sum_{J \in \mathcal{M}_{\mathrm{deep}}(I_r)} |J^*|_\sigma \leq \left(\mathfrak{T}^{\Omega \mathcal{Q}^n}_{(\mathbf{R}_\Psi^{\alpha,n})_1}\right)^2 \sum_{r=1}^\infty \beta |I_r|_\sigma \leq \beta \left(\mathfrak{T}^{\Omega \mathcal{Q}^n}_{(\mathbf{R}_\Psi^{\alpha,n})_1}\right)^2 |I|_\sigma . \end{aligned}$$

This leaves us to consider the term

$$\begin{aligned} A_3 &= \sum_{J \in \mathcal{M}_{\mathrm{deep}}} \frac{1}{|J|_\omega} \int_{J \cap L} \int_{J \cap L} \left\{(\mathbf{R}_\Psi^{\alpha,n})_1 \left(\mathbf{1}_{\mathrm{S}(J^*)} \sigma\right) (x^1, 0') - (\mathbf{R}_\Psi^{\alpha,n})_1 \left(\mathbf{1}_{\mathrm{S}(J^*)} \sigma\right) (z^1, 0')\right\}^2 d\omega(x)\, d\omega(z) \\ &= 2 \sum_{J \in \mathcal{M}_{\mathrm{deep}}} \int_{J \cap L} \left\{(\mathbf{R}_\Psi^{\alpha,n})_1 \left(\mathbf{1}_{\mathrm{S}(J^*)} \sigma\right) (x^1, 0') - \mathbb{E}^\omega_{J \cap L} \left[(\mathbf{R}_\Psi^{\alpha,n})_1 \left(\mathbf{1}_{\mathrm{S}(J^*)} \sigma\right) (z^1, 0')\right]\right\}^2 d\omega(x) , \end{aligned}$$

in which we do *not* discard the difference. However, because the average is subtracted off, we can apply the Quasienergy Lemma 24, together with duality

$$\begin{aligned} \left\|\mathbf{R}_\Psi^{\alpha,n}(\nu) - \mathbb{E}^\omega_J \mathbf{R}_\Psi^{\alpha,n}(\nu)\right\|_{L^2(\omega)} &= \sup_{\|\Psi_J\|_{L^2(\omega)}=1} \left|\left\langle \mathbf{R}_\Psi^{\alpha,n}(\nu) - \mathbb{E}^\omega_J \mathbf{R}_\Psi^{\alpha,n}(\nu), \Psi_J \right\rangle_\omega\right| \\ &= \sup_{\|\Psi_J\|_{L^2(\omega)}=1} \left|\left\langle \mathbf{R}_\Psi^{\alpha,n}(\nu), \Psi_J \right\rangle_\omega\right| , \end{aligned}$$

to each term in this sum to dominate it by,

$$B = \sum_{J \in \mathcal{M}_{\mathrm{deep}}} \left(\frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_{\mathrm{S}(J^*)} \sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}^\omega_J \mathbf{x}\right\|^2_{L^2(\omega)} . \tag{3.20}$$

To estimate B, we first assume that $n-1 \leq \alpha < n$ so that $\mathrm{P}^\alpha\left(J, \mathbf{1}_{\mathrm{S}(J^*)} \sigma\right) \leq \mathcal{P}^\alpha\left(J, \mathbf{1}_{\mathrm{S}(J^*)} \sigma\right)$, and then use $\left\|\mathsf{P}^\omega_J \mathbf{x}\right\|^2_{L^2(\omega)} \lesssim |J|^{\frac{2}{n}} |J|_\omega$ and apply the $\mathcal{A}_2^\alpha$ condition

with holes to obtain the following 'pivotal reversal' of quasienergy,

$$\begin{aligned}
B &\lesssim \sum_{J\in\mathcal{M}_{\mathrm{deep}}} \mathrm{P}^{\alpha}\left(J,\mathbf{1}_{\mathrm{S}(J^*)}\sigma\right)^2 |J|_{\omega} \leq \sum_{J\in\mathcal{M}_{\mathrm{deep}}} \mathrm{P}^{\alpha}\left(J,\mathbf{1}_{\mathrm{S}(J^*)}\sigma\right)\left\{\mathcal{P}^{\alpha}\left(J,\mathbf{1}_{\mathrm{S}(J^*)}\sigma\right)|J|_{\omega}\right\} \\
&\leq \mathcal{A}_2^{\alpha} \sum_{J\in\mathcal{M}_{\mathrm{deep}}} \mathrm{P}^{\alpha}\left(J,\mathbf{1}_{\mathrm{S}(J^*)}\sigma\right)|J|^{1-\frac{\alpha}{n}} = \mathcal{A}_2^{\alpha} \sum_{J\in\mathcal{M}_{\mathrm{deep}}} \int_{\mathrm{S}(J^*)} \frac{|J|^{\frac{1}{n}}\,|J|^{1-\frac{\alpha}{n}}}{\left(|J|^{\frac{1}{n}}+|y-c_J|\right)^{n+1-\alpha}} d\sigma(y) \\
&= \mathcal{A}_2^{\alpha} \sum_{J\in\mathcal{M}_{\mathrm{deep}}} \int_{\mathrm{S}(J^*)} \left(\frac{|J|^{\frac{1}{n}}}{|J|^{\frac{1}{n}}+|y-c_J|}\right)^{n+1-\alpha} d\sigma(y) \\
&= \mathcal{A}_2^{\alpha} \int_I \left\{\sum_{J\in\mathcal{M}_{\mathrm{deep}}} \left(\frac{|J|^{\frac{1}{n}}}{|J|^{\frac{1}{n}}+|y-c_J|}\right)^{n+1-\alpha} \mathbf{1}_{\mathrm{S}(J^*)}(y)\right\} d\sigma(y) \\
&\equiv \mathcal{A}_2^{\alpha} \int_I F(y)\, d\sigma(y).
\end{aligned}$$

At this point we claim that $F(y) \leq C$ with a constant C independent of the decomposition $\mathcal{M}_{\mathrm{deep}} = \dot{\bigcup_{r\geq 1}} \mathcal{M}_{\mathrm{deep}}(I_r)$. Indeed, if y is fixed, then the quasicubes $J \in \mathcal{M}_{\mathrm{deep}}$ for which $y \in \mathrm{S}(J^*)$ satisfy

$$J \cap \mathrm{Sh}(y;\gamma) \neq \emptyset, \tag{3.21}$$

where $\mathrm{Sh}(y;\gamma)$ is the Carleson shadow of the point y onto the x_1-axis L, defined as the interval on L with length $\frac{1}{5}\gamma\,\mathrm{dist}(y,L)$ and center equal to the point on L that is closest to y. If a quasicube J intersects $\mathrm{Sh}(y;\gamma)$, and $y \in \mathrm{S}(J^*)$, we must have

$$\ell(J) \leq C_0\,\mathrm{dist}(y,L),$$

where $C_0 = C_0(\gamma, R_{\mathrm{big}})$ is a positive constant depending on γ and the comparability constant R_{big} for Ω appearing in (2.1). We have thus shown that $J \in \mathcal{M}_{\mathrm{deep}}$ and $y \in \mathrm{S}(J^*)$ imply $J \cap L \subset C\,\mathrm{Sh}(y;\gamma)$ with $C = 2C_0 + 1$.

Let $\mathcal{J} = J \cap L$ be the intersection of the quasicube J with L, and note that the linear measure of $\mathcal{J}$ satisfies $|\mathcal{J}| \lesssim \ell(J)$. Moreover, $\mathcal{J}$ need not be an interval if the quasicube's edge is close to being parallel to L. Fix a point y. Then for a quasicube J satisfying both (3.21) and $y \in \mathrm{S}(J^*)$, the set $\mathcal{J}$ is contained inside the multiple $C\,\mathrm{Sh}(y;\gamma)$ of the shadow, and

$$|y - c_J| \gtrsim \mathrm{dist}(y,L).$$

Now we face two difficulties that do not arise for usual cubes with a side parallel to L. First, as already mentioned, $\mathcal{J}$ need not be an interval, and in fact may be a quite complicated set, and second that a quasicube may intersect the line L in a set having linear measure far less than its side length, for example when a tilted cube intersects L near a vertex. Both of these difficulties are surmounted using the fact that the quasicubes J belong to a collection $\mathcal{M}_{\mathbf{r}-\mathrm{deep}}(I_r)$ for some r (we are still suppressing the index ℓ). Indeed, we first show that there is a positive constant C' such that for each r we have

$$\sum_{\substack{J\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}(I_r) \\ \emptyset\neq\mathcal{J}\subset C\,\mathrm{Sh}(y;\gamma),\, y\in\mathrm{S}(J^*)}} \ell(J) \leq \beta\left|I_r \cap C'\,\mathrm{Sh}(y;\gamma)\right|, \tag{3.22}$$

where β is the constant appearing in the bounded overlap condition (2.4). To see this, we note that if $\emptyset \neq \mathcal{J} = J \cap L$, then $J^* = \gamma J$ satisfies $|J^* \cap L| \geq \ell(J)$ provided γ is large enough depending on the constant R_{big} in (2.1), and we also have $J^* \subset C' \operatorname{Sh}(y;\gamma)$ where $C' = C'(C, \gamma, R_{\text{big}})$ is a positive constant depending on C, γ and R_{big}. Altogether we thus have

$$\begin{aligned}
\sum_{\substack{J \in \mathcal{M}_{\mathbf{r}-\text{deep}}(I_r) \\ \emptyset \neq \mathcal{J} \subset C \operatorname{Sh}(y;\gamma), y \in \mathrm{S}(J^*)}} \ell(J) &\leq \sum_{\substack{J \in \mathcal{M}_{\mathbf{r}-\text{deep}}(I_r) \\ \emptyset \neq \mathcal{J} \subset C \operatorname{Sh}(y;\gamma), y \in \mathrm{S}(J^*)}} |J^* \cap L| = \int_L \left(\sum_{\substack{J \in \mathcal{M}_{\mathbf{r}-\text{deep}}(I_r) \\ \emptyset \neq \mathcal{J} \subset C \operatorname{Sh}(y;\gamma), y \in \mathrm{S}(J^*)}} 1_{J^*} \right) dx \\
&\leq \beta \int_{L \cap I_r} \mathbf{1}_{C' \operatorname{Sh}(y;\gamma)} dx \leq \beta \int_{C' \operatorname{Sh}(y;\gamma) \cap I_r} dx = \beta \left| C' \operatorname{Sh}(y;\gamma) \cap I_r \right|,
\end{aligned}$$

which proves (3.22).

Now we continue with the estimate

$$\begin{aligned}
(3.23) \quad & \\
\sum_{\substack{J \in \mathcal{M}_{\text{deep}} \\ \mathcal{J} \subset C \operatorname{Sh}(y;\gamma), y \in \mathrm{S}(J^*)}} \left(\frac{|J|^{\frac{1}{n}}}{|J|^{\frac{1}{n}} + |y - c_J|} \right)^{n+1-\alpha} &= \sum_{r=1}^{\infty} \sum_{\substack{J \in \mathcal{M}_{\mathbf{r}-\text{deep}}(I_r) \\ \emptyset \neq \mathcal{J} \subset C \operatorname{Sh}(y;\gamma), y \in \mathrm{S}(J^*)}} \left(\frac{|J|^{\frac{1}{n}}}{|J|^{\frac{1}{n}} + |y - c_J|} \right)^{n+1-\alpha} \\
\lesssim \sum_{r=1}^{\infty} \sum_{\substack{J \in \mathcal{M}_{\mathbf{r}-\text{deep}}(I_r) \\ \emptyset \neq \mathcal{J} \subset C \operatorname{Sh}(y;\gamma), y \in \mathrm{S}(J^*)}} \left(\frac{|J|^{\frac{1}{n}}}{|y - c_J|} \right)^{n-\alpha} \frac{|J|^{\frac{1}{n}}}{\operatorname{dist}(y, L)} &\lesssim \frac{1}{\operatorname{dist}(y, L)} \sum_{r=1}^{\infty} \left\{ \sum_{\substack{J \in \mathcal{M}_{\mathbf{r}-\text{deep}}(I_r) \\ \emptyset \neq \mathcal{J} \subset C \operatorname{Sh}(y;\gamma), y \in \mathrm{S}(J^*)}} |J|^{\frac{1}{n}} \right\} \\
\lesssim \frac{1}{\operatorname{dist}(y, L)} \sum_{r=1}^{\infty} \beta \left| I_r \cap C' \operatorname{Sh}(y;\gamma) \right| &\lesssim \beta \frac{1}{\operatorname{dist}(y, L)} \left| C' \operatorname{Sh}(y;\gamma) \right| \lesssim \beta\gamma,
\end{aligned}$$

because $n - \alpha > 0$ and the sets $\{I_r \cap C' \operatorname{Sh}(y;\gamma)\}_{r=1}^{\infty}$ are pairwise disjoint in $C' \operatorname{Sh}(y;\gamma)$. It is here that the one-dimensional nature of ω permits the summing of the side lengths of the quasicubes. Thus we have

$$B \leq \mathcal{A}_2^{\alpha} \int_I F(y)\, d\sigma(y) \leq C \mathcal{A}_2^{\alpha} |I|_{\sigma} \ ,$$

which is the desired estimate in the case that $n - 1 \leq \alpha < n$.

Now we suppose that $0 \leq \alpha < n-1$ and use Cauchy-Schwarz to obtain

$$\begin{aligned}
\mathrm{P}^{\alpha}\left(J, \mathbf{1}_{\mathrm{S}(J^*)}\sigma\right) &= \int_{\mathrm{S}(J^*)} \frac{|J|^{\frac{1}{n}}}{\left(|J|^{\frac{1}{n}} + |y-c_J|\right)^{n+1-\alpha}} d\sigma(y) \\
&\leq \left\{\int_{\mathrm{S}(J^*)} \frac{|J|^{\frac{1}{n}}}{\left(|J|^{\frac{1}{n}} + |y-c_J|\right)^{n+1-\alpha}} \left(\frac{|J|^{\frac{1}{n}}}{|J|^{\frac{1}{n}} + |y-c_J|}\right)^{n-1-\alpha} d\sigma(y)\right\}^{\frac{1}{2}} \\
&\quad \times \left\{\int_{\mathrm{S}(J^*)} \frac{|J|^{\frac{1}{n}}}{\left(|J|^{\frac{1}{n}} + |y-c_J|\right)^{n+1-\alpha}} \left(\frac{|J|^{\frac{1}{n}}}{|J|^{\frac{1}{n}} + |y-c_J|}\right)^{\alpha+1-n} d\sigma(y)\right\}^{\frac{1}{2}} \\
&= \mathcal{P}^{\alpha}\left(J, \mathbf{1}_{\mathrm{S}(J^*)}\sigma\right)^{\frac{1}{2}} \\
&\quad \times \left\{\int_{\mathrm{S}(J^*)} \frac{\left(|J|^{\frac{1}{n}}\right)^{\alpha+2-n}}{\left(|J|^{\frac{1}{n}} + |y-c_J|\right)^{2}} d\sigma(y)\right\}^{\frac{1}{2}}.
\end{aligned}$$

Then arguing as above we have

(3.24)

$$\begin{aligned}
B &\leq \sum_{J \in \mathcal{M}_{\mathrm{deep}}} \mathrm{P}^{\alpha}\left(J^*, \mathbf{1}_{\mathrm{S}(J^*)}\sigma\right)^2 |J|_{\omega} \\
&\leq \sum_{J \in \mathcal{M}_{\mathrm{deep}}} \left\{\mathcal{P}^{\alpha}\left(J^*, \mathbf{1}_{\mathrm{S}(J^*)}\sigma\right) |J|_{\omega}\right\} \int_{\mathrm{S}(J^*)} \frac{\left(|J|^{\frac{1}{n}}\right)^{\alpha+2-n}}{\left(|J|^{\frac{1}{n}} + |y-c_J|\right)^{2}} d\sigma(y) \\
&\leq \mathcal{A}_2^{\alpha} \sum_{J \in \mathcal{M}_{\mathrm{deep}}} |J|^{1-\frac{\alpha}{n}} \int_{\mathrm{S}(J^*)} \frac{\left(|J|^{\frac{1}{n}}\right)^{\alpha+2-n}}{\left(|J|^{\frac{1}{n}} + |y-c_J|\right)^{2}} d\sigma(y) \\
&= \mathcal{A}_2^{\alpha} \sum_{J \in \mathcal{M}_{\mathrm{deep}}} \int_{\mathrm{S}(J^*)} \frac{|J|^{\frac{2}{n}}}{\left(|J|^{\frac{1}{n}} + |y-c_J|\right)^{2}} d\sigma(y) \\
&= \mathcal{A}_2^{\alpha} \int_I \left\{\sum_{J \in \mathcal{M}_{\mathrm{deep}}} \left(\frac{|J|^{\frac{1}{n}}}{|J|^{\frac{1}{n}} + |y-c_J|}\right)^2 \mathbf{1}_{\mathrm{S}(J^*)}(y)\right\} d\sigma(y) \\
&\equiv \mathcal{A}_2^{\alpha} \int_I F(y)\, d\sigma(y),
\end{aligned}$$

and again $F(y) \leq C$ is the calculation above when $n+1-\alpha$ is replaced by 2:

$$\begin{aligned}
\sum_{\substack{J\in\mathcal{M}_{\mathrm{deep}}\\ \mathcal{J}\subset C\,\mathrm{Sh}(y;\gamma),y\in\mathrm{S}(J^*)}} \left(\frac{|J|^{\frac{1}{n}}}{|J|^{\frac{1}{n}}+|y-c_J|}\right)^2 &= \sum_{r=1}^{\infty} \sum_{\substack{J\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}(I_r)\\ \emptyset\neq\mathcal{J}\subset C\,\mathrm{Sh}(y;\gamma),y\in\mathrm{S}(J^*)}} \left(\frac{|J|^{\frac{1}{n}}}{|J|^{\frac{1}{n}}+|y-c_J|}\right)^2 \\
&\lesssim \sum_{r=1}^{\infty} \sum_{\substack{J\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}(I_r)\\ \emptyset\neq\mathcal{J}\subset C\,\mathrm{Sh}(y;\gamma),y\in\mathrm{S}(J^*)}} \frac{|J|^{\frac{1}{n}}}{|y-c_J|}\frac{|J|^{\frac{1}{n}}}{\mathrm{dist}(y,L)} \\
&\lesssim \frac{1}{\mathrm{dist}(y,L)} \sum_{r=1}^{\infty} \left\{ \sum_{\substack{J\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}(I_r)\\ \emptyset\neq\mathcal{J}\subset C\,\mathrm{Sh}(y;\gamma),y\in\mathrm{S}(J^*)}} |J|^{\frac{1}{n}} \right\} \\
&\lesssim \frac{1}{\mathrm{dist}(y,L)} \sum_{r=1}^{\infty} \beta\,|I_r\cap C'\,\mathrm{Sh}(y;\gamma)| \\
&\lesssim \beta\frac{1}{\mathrm{dist}(y,L)}\,|C'\,\mathrm{Sh}(y;\gamma)| \lesssim \beta\gamma.
\end{aligned}$$

Thus we again have

$$B \leq \mathcal{A}_2^\alpha \int_I F(y)\,d\sigma(y) \leq C\mathcal{A}_2^\alpha\,|I|_\sigma\ ,$$

and this completes the proof of necessity of the quasienergy conditions when one of the measures is supported on a line.

3.3. **Backward triple testing and quasiweak boundedness property.** In this subsection we show that for a measure supported on a line, the backward triple quasitesting condition,

$$\int_{3Q'} \left|\mathbf{R}_\Psi^{\alpha,n}(1_{Q'}\omega)\right|^2 d\sigma \leq \mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{triple},\mathrm{dual}}\,|Q'|_\omega\ , \tag{3.25}$$

is controlled by the $\mathcal{A}_2^\alpha$ conditions with holes and the two quasitesting conditions, namely

$$\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{triple},\mathrm{dual}} \lesssim \mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{dual}} + \sqrt{\mathcal{A}_2^\alpha} + \sqrt{\mathcal{A}_2^{\alpha,\mathrm{dual}}},$$

provided that Ω is a C^1 diffeomorphism and L-transverse, where L is the support of ω. It is then an easy consequence of the Cauchy-Schwarz inequality that the weak boundedness property is also controlled by the A_2 conditions and the two quasitesting conditions,

$$\int_Q \mathbf{R}_\Psi^{\alpha,n}(1_{Q'}\omega)\,d\sigma \lesssim \left(\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{dual}} + \sqrt{\mathcal{A}_2^\alpha} + \sqrt{\mathcal{A}_2^{\alpha,\mathrm{dual}}}\right)\sqrt{|Q|_\sigma\,|Q'|_\omega},$$

for all pairs of quasicubes Q and Q' of size comparable to their distance apart.

We will use the following two properties of an L-transverse C^1 diffeomorphism Ω as defined in Definition 12 above.

Lemma 26. *Suppose that $\Omega:\mathbb{R}^n\to\mathbb{R}^n$ is a C^1 diffeomorphism and L-transverse. Then if $\mathbf{e}_L$ is a unit vector in the direction of L, we have*

$$\left|\left\langle \frac{D\Omega^{-1}(x)\,\mathbf{e}_L}{|D\Omega^{-1}(x)\,\mathbf{e}_L|},\mathbf{e}_k\right\rangle\right| \leq \frac{1+\eta}{2}, \qquad \text{for } x\in\mathbb{R}^n \text{ and } 1\leq k\leq n, \tag{3.26}$$

and

$$Q \cap L \text{ is connected whenever } Q \in \Omega \mathcal{P}^n. \tag{3.27}$$

Proof. Choose a rotation $R \in \digamma_{\mathbf{e}_L,\eta}$. Then we have

$$\begin{aligned}\left|\left\langle \frac{D\Omega^{-1}(x)\,\mathbf{e}_L}{|D\Omega^{-1}(x)\,\mathbf{e}_L|} - D\Omega^{-1}(x)\,\mathbf{e}_L, \mathbf{e}_k \right\rangle\right| &\leq \left|\frac{D\Omega^{-1}(x)\,\mathbf{e}_L}{|D\Omega^{-1}(x)\,\mathbf{e}_L|} - D\Omega^{-1}(x)\,\mathbf{e}_L\right| \\ &\leq \left|R\mathbf{e}_L - D\Omega^{-1}(x)\,\mathbf{e}_L\right|\end{aligned}$$

since $\left|\frac{v}{|v|} - v\right| = \operatorname{dist}\left(v, \mathbb{S}^{n-1}\right) \leq |v - R\mathbf{e}_L|$. Thus using $\left\|D\Omega^{-1} - R\right\|_\infty < \frac{1-\eta}{4}$ from the definition of L-transverse, we obtain

$$\begin{aligned}&\left|\left\langle \frac{D\Omega^{-1}(x)\,\mathbf{e}_L}{|D\Omega^{-1}(x)\,\mathbf{e}_L|}, \mathbf{e}_k \right\rangle\right| \\ = &\left|\left\langle \frac{D\Omega^{-1}(x)\,\mathbf{e}_L}{|D\Omega^{-1}(x)\,\mathbf{e}_L|} - D\Omega^{-1}(x)\,\mathbf{e}_L, \mathbf{e}_k \right\rangle + \langle R\mathbf{e}_L, \mathbf{e}_k\rangle + \left\langle \left(D\Omega^{-1}(x) - R\right)\mathbf{e}_L, \mathbf{e}_k\right\rangle\right| \\ \leq &\left\|D\Omega^{-1} - R\right\|_\infty + \eta + \left\|D\Omega^{-1} - R\right\|_\infty < \frac{1+\eta}{2},\end{aligned}$$

which proves (3.26).

Now suppose that $Q \in \Omega\mathcal{P}^n$ satisfies $Q \cap L \neq \emptyset$. Then $\Omega^{-1}L$ is the image of a differentiable curve. Let $\varphi : \mathbb{R} \to \mathbb{R}^n$ be a parameterization of $\Omega^{-1}L$. The tangent directions $\frac{D\varphi(t)}{|D\varphi(t)|}$ of the curve $\Omega^{-1}L$ are given by $\frac{D\Omega^{-1}(\varphi(t))\mathbf{e}_L}{|D\Omega^{-1}(\varphi(t))\mathbf{e}_L|}$, which satisfy (3.26). Set $K \equiv \Omega^{-1}Q$ and note that K is an ordinary half open half closed cube in $\mathcal{P}^n$, which without loss of generality we may take to be $K = [-1,1)^n$. Let $\alpha \equiv \inf \varphi^{-1}K$ and $\beta \equiv \sup \varphi^{-1}K$. Now assume in order to derive a contradiction that $\Omega^{-1}L \cap \Omega^{-1}Q$ is not connected. It follows from (3.26) that if $\varphi(t) \in \partial K$, then the tangent line at $\varphi(t)$ intersects the complement of $\overline{K}$ in any neighbourhood of $\varphi(t)$, and hence there is $t_0 \in (\alpha,\beta)$ such that $\varphi(t_0) = (\varphi_1(t_0), ..., \varphi_n(t_0)) \notin \overline{K}$. Thus there is k_2 such that $\left|\varphi_{k_2}(t_0)\right| > 1$. Let k_1 be any index other than k_2.

Let P be orthogonal projection of $\mathbb{R}^n$ onto the 2-plane Π spanned by $\mathbf{e}_{k_1}$ and $\mathbf{e}_{k_2}$. Then $\mathsf{P}\varphi$ is a differentiable curve whose image lies in Π and satisfies the following analogue of (3.26):

$$\left|\left\langle \frac{D\mathsf{P}\varphi(t)}{|D\mathsf{P}\varphi(t)|}, \mathbf{e}_k \right\rangle\right| = \left|\left\langle \frac{D\mathsf{P}\Omega^{-1}(\varphi(t))\,\mathbf{e}_L}{|D\mathsf{P}\Omega^{-1}(\varphi(t))\,\mathbf{e}_L|}, \mathbf{e}_k \right\rangle\right| \leq \frac{1+\eta}{2} < 1, \quad \text{for } t \in \mathbb{R} \text{ and } k = k_1, k_2\,. \tag{3.28}$$

Thus the image $\mathsf{P}\Omega^{-1}L$ of the curve $\mathsf{P}\varphi$ may be written as the graph of a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ whose domain is identified with the x_{k_1}-axis and whose range is identified with the x_{k_2}-axis. Then we have

$$|f(x_{k_1})| = \left|\varphi_{k_2}(t_0)\right| > 1$$

for $x_{k_1} = \varphi_{k_1}(t_0)$. The map $g(t) = f^{-1}\left(\varphi_{k_2}(t)\right)$ is differentiable and one-to-one, hence monotone and $g(\alpha), g(\beta) \in [-1,1]$. We may suppose g is strictly increasing. We also have $f(g(\alpha)) = \varphi_{k_2}(\alpha), f(g(\beta)) = \varphi_{k_2}(\beta) \in [-1,1]$ since $\varphi(\alpha), \varphi(\beta) \in \overline{K}$. It follows that $x_{k_1} \in (g(\alpha), g(\beta))$, and hence f must have a relative extreme value at some point $z \in (g(\alpha), g(\beta))$. But then $f'(z) = 0$ which implies $D\mathsf{P}\varphi\left(g^{-1}(z)\right)$ is parallel to $\mathbf{e}_{k_1}$, and so contradicts $\left|\left\langle \frac{D\mathsf{P}\varphi\left(g^{-1}(z)\right)}{|D\mathsf{P}\varphi(g^{-1}(z))|}, \mathbf{e}_{k_1} \right\rangle\right| < 1$ from (3.28). □

To prove the backward triple quasitesting condition in (3.25),

$$\int_{3Q'} \left|\mathbf{R}_{\Psi}^{\alpha,n}\left(1_{Q'}\omega\right)\right|^2 d\sigma \leq \mathfrak{T}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{triple},\mathrm{dual}} \left|Q'\right|_{\omega} \ ,$$

it suffices to decompose the triple $3Q'$ into 3^n dyadic quasicubes Q of side length that of Q', and then apply backward testing to $Q=Q'$ which gives $\mathfrak{T}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{dual}} \left|Q'\right|_{\omega}$, and then to prove

$$\int_{Q} \left|\mathbf{R}_{\Psi}^{\alpha,n}\left(1_{Q'}\omega\right)\right|^2 d\sigma \leq \left(\sqrt{\mathcal{A}_2^{\alpha}}+\sqrt{\mathcal{A}_2^{\alpha,\mathrm{dual}}}\right) \left|Q'\right|_{\omega} \ ,$$

where Q and Q' are distinct quasicubes of equal side length in a common dyadic quasigrid that share an $(n-1)$-dimensional quasiface $\mathbb{F}$ in common. We also assume that ω is supported on a line L that is parallel to a coordinate axis. The cases when the quasicubes Q and Q' meet in an 'edge' of smaller dimension, is handled in similar fashion.

From Lemma 26 we see that the line L meets the quasihyperplane $\mathbb{H}$ containing the quasiface $\mathbb{F}$ at an angle at least $\varepsilon>0$, and that the intersection $Q'\cap L$ is an interval. Now select the smallest possible dyadic subquasicube Q'' of Q' such that $Q'\cap L=Q''\cap L$ to obtain that the length of the interval $Q''\cap L$ is comparable to $\ell\left(Q''\right)$. Then since $Q\setminus 3Q''$ is well separated from Q'' we have

$$\int_{Q\setminus 3Q''} \left|\mathbf{R}_{\Psi}^{\alpha,n}\left(1_{Q''}\omega\right)\right|^2 d\sigma \leq \left(\sqrt{\mathcal{A}_2^{\alpha}}+\sqrt{\mathcal{A}_2^{\alpha,\mathrm{dual}}}\right) \left|Q''\right|_{\omega} \ ,$$

and it remains to consider the integrals $\int_{Q'''} \left|\mathbf{R}_{\Psi}^{\alpha,n}\left(1_{Q''}\omega\right)\right|^2 d\sigma$ as Q''' ranges over all dyadic quasicubes in $3Q''\cap Q$ with side length $\ell\left(Q''\right)$. Now we relabel Q''' and Q'' as Q and Q', and then without loss of essential generality, we may assume that with R denoting an appropriate rotation, we have

(1) $Q=\Psi K$ where $K=R\left(\left[-1,0\right]\times\left[-\frac{1}{2},\frac{1}{2}\right]^{n-1}\right)$,
(2) $Q'=\Psi K'$ where $K'=R\left(\left[0,1\right]\times\left[-\frac{1}{2},\frac{1}{2}\right]^{n-1}\right)$,
(3) $\operatorname{supp}\omega\subset L=\left(-\infty,\infty\right)\times\left\{\left(0,...,0\right)\right\}$ the x_1 -axis
(4) $Q'\cap L$ is an interval of length comparable to $\ell\left(Q'\right)$.

Then the restriction $\omega_{Q'}$ of ω to the quasicube Q' has support $\operatorname{supp}\omega_{Q'}$ contained in the line segment $S\equiv Q'\cap L$, while the restriction σ_Q of σ to the quasicube Q has support $\operatorname{supp}\sigma_Q$ contained in the quasicube Q. We exploit the distinguished role played by the unique point in $\partial Q\cap\partial Q'\cap L$, which we relabel as the origin, by writing $y=t\xi\in Q$ where $t=\left|y\right|$ and $\xi\in\mathbb{S}^{n-1}$, and by writing $x=\left(s,\mathbf{0}\right)\in Q'\cap L$, so that for an appropriate $a\approx 1$, we have

$$\begin{aligned}
&\int_{Q} \left|\mathbf{R}_{\Psi}^{\alpha,n}\left(1_{Q'}\omega\right)\right|^2 d\sigma \lesssim \int_{Q}\left\{\int_{Q'}\left|y-x\right|^{\alpha-n}d\omega\left(x\right)\right\}^2 d\sigma\left(y\right)\\
=\ &\int_{Q}\left\{\int_0^a\left|t\xi-\left(s,\mathbf{0}\right)\right|^{\alpha-n}d\omega\left(s,\mathbf{0}\right)\right\}^2 d\sigma\left(t\xi\right)\approx\int_{Q}\left\{\int_0^a\left(t+s\right)^{\alpha-n}d\omega\left(s,\mathbf{0}\right)\right\}^2 d\sigma\left(t\xi\right)\\
\equiv\ &\int_0^{\infty}\left\{\int_0^{\infty}\left(t+s\right)^{\alpha-n}d\widetilde{\omega}\left(s\right)\right\}^2 d\widetilde{\sigma}\left(t\right)=\int_0^{\infty}\left\{\left(\int_0^t+\int_t^{\infty}\right)\left(t+s\right)^{\alpha-n}d\widetilde{\omega}\left(s\right)\right\}^2 d\widetilde{\sigma}\left(t\right)\\
\approx\ &\int_0^{\infty}\left\{\int_0^t d\widetilde{\omega}\left(s\right)\right\}^2 t^{2\alpha-2n}d\widetilde{\sigma}\left(t\right)+\int_0^{\infty}\left\{\int_t^{\infty}s^{\alpha-n}d\widetilde{\omega}\left(s\right)\right\}^2 d\widetilde{\sigma}\left(t\right)\equiv I+II,
\end{aligned}$$

where the one dimensional measures $\widetilde{\omega}$ and $\widetilde{\sigma}$ are uniquely determined by ω, Q' and σ, Q respectively by the passage from the second line to the third line above. Note also that the approximation in the second line above follows from (3.26). Now as in [LaSaShUr2], we apply Muckenhoupt's two weight Hardy inequality for general measures (see Hytönen [Hyt2] for a proof), to obtain

$$\int_0^\infty \left\{ \int_{(0,t]} f(s)\, d\mu(s) \right\}^2 d\nu(t) \lesssim \left\{ \sup_{0<r<\infty} \left(\int_{[r,\infty)} d\nu \right) \left(\int_{(0,r]} d\mu \right) \right\} \int_0^\infty f(s)^2\, d\mu(s),$$

with $\mu = \widetilde{\omega}$, $d\nu(t) = t^{2\alpha-2n} d\widetilde{\sigma}(t)$ and $f = 1$ to obtain that

$$\begin{aligned} I &= \int_0^\infty \left\{ \int_{(0,t]} d\widetilde{\omega}(s) \right\}^2 t^{2\alpha-2n} d\widetilde{\sigma}(t) \\ &\lesssim \left\{ \sup_{0<r<\infty} \left(\int_{[r,\infty)} t^{2\alpha-2n} d\widetilde{\sigma}(t) \right) \left(\int_{(0,r]} d\widetilde{\omega}(s) \right) \right\} \int_0^\infty d\widetilde{\omega}(s) \lesssim \mathcal{A}_2^\alpha \ \left|Q'\right|_\omega \end{aligned}$$

since $\int_0^\infty d\widetilde{\omega}(s) = \left|Q'\right|_\omega$ and

$$\begin{aligned} & \left(\int_{[r,\infty)} t^{2\alpha-2n} d\widetilde{\sigma}(t) \right) \left(\int_{(0,r]} d\widetilde{\omega}(s) \right) \\ \approx & \int_{Q \cap \{|y| \geq r\}} |y-x|^{2\alpha-2n}\, d\sigma(y) \int_{Q' \cap \{|x| \leq r\}} d\omega(x) \\ \approx & \int_{Q \cap \{|y| \geq r\}} \left(\frac{r}{(|y|+r)^2} \right)^{n-\alpha} d\sigma(y) \ \ r^{\alpha-n} \left|Q' \cap \{|x| \leq r\}\right|_\omega \lesssim \mathcal{A}_2^\alpha\ . \end{aligned}$$

Then we apply the two weight dual Hardy inequality

$$\int_0^\infty \left\{ \int_{[t,\infty)} f(s)\, d\mu(s) \right\}^2 d\nu(t) \leq \left\{ \sup_{0<r<\infty} \left(\int_{(0,r]} d\nu \right) \left(\int_{[r,\infty)} d\mu \right) \right\} \int_0^\infty f(s)^2\, d\mu(s),$$

with $d\mu(s) = s^{2\alpha-2n} d\widetilde{\omega}(s)$, $d\nu(t) = d\widetilde{\sigma}(t)$ and $f(s) = s^{n-\alpha}$ to obtain that

$$\begin{aligned} II &= \int_0^\infty \left\{ \int_{[t,\infty)} s^{\alpha-n} d\widetilde{\omega}(s) \right\}^2 d\widetilde{\sigma}(t) = \int_0^\infty \left\{ \int_{[s,\infty)} s^{n-\alpha} d\mu(s) \right\}^2 d\widetilde{\sigma}(t) \\ &\lesssim \left\{ \sup_{0<r<\infty} \left(\int_{(0,r]} d\widetilde{\sigma}(t) \right) \left(\int_{[r,\infty)} s^{2\alpha-2n} d\widetilde{\omega}(s) \right) \right\} \int_0^\infty s^{2n-2\alpha} d\mu(s) \lesssim \mathcal{A}_2^\alpha \ \left|Q'\right|_\omega \end{aligned}$$

since $\int_0^\infty s^{2n-2\alpha} d\mu(s) = \int_0^\infty d\widetilde{\omega}(s) = \left|Q'\right|_\omega$ and $\left(\int_{(0,r]} d\widetilde{\sigma}(t) \right) \left(\int_{[r,\infty)} s^{2\alpha-2n} d\widetilde{\omega}(s) \right) \lesssim \mathcal{A}_2^{\alpha,\mathrm{dual}}$ just as above.

Remark 27. *In the case where one measure is supported on the x-axis, we need only test over cubes with sides parallel to the coordinate axes. For the Cauchy operator this is in* [LaSaShUrWi] *and for the higher dimensional case see the earlier versions of the current paper on the arXiv.*

4. One measure compactly supported on a $C^{1,\delta}$ curve

Suppose that $\Psi:\mathbb{R}^n\rightarrow\mathbb{R}^n$ is a $C^{1,\delta}$ diffeomorphism. Recall the associated class of conformal Riesz vector transforms $\mathbf{R}_{\Psi}^{\alpha,n}$ whose kernels $\mathbf{K}_{\Psi}^{\alpha,n}$ are given by

$$\mathbf{K}_{\Psi}^{\alpha,n}\left(x,y\right)=\frac{y-x}{\left|\Psi\left(y\right)-\Psi\left(x\right)\right|^{n+1-\alpha}}.$$

First we investigate the effect of a change of variable on the statement of the $T1$ theorem.

4.1. **Changes of variable.** Suppose that $\Psi:\mathbb{R}^n\rightarrow\mathbb{R}^n$ is a $C^{1,\delta}$ diffeomorphism, i.e. that both Ψ and its inverse Ψ^{-1} are globally $C^{1,\delta}$ maps. In particular we have that

$$\begin{aligned}
&\left|\Psi\left(y\right)-\Psi\left(x\right)\right|\leq\left\|\Psi\right\|_{C^1}\left|y-x\right|, \qquad (4.1)\\
&\left|\Psi^{-1}\left(w\right)-\Psi^{-1}\left(z\right)\right|\leq\left\|\Psi^{-1}\right\|_{C^1}\left|w-z\right|,\\
\Longrightarrow\quad &\frac{1}{\left\|\Psi^{-1}\right\|_{C^1}}\leq\frac{\left|\Psi\left(y\right)-\Psi\left(x\right)\right|}{\left|y-x\right|}=\frac{\left|w-z\right|}{\left|\Psi^{-1}\left(w\right)-\Psi^{-1}\left(z\right)\right|}\leq\left\|\Psi\right\|_{C^1},\\
\Longrightarrow\quad &\frac{1}{\left\|\Psi^{-1}\right\|_{C^1}}\leq\left\|D\Psi\right\|_{\infty}\leq\left\|\Psi\right\|_{C^1}\ .
\end{aligned}$$

Let

$$\Psi^{*}\mathbf{K}^{\alpha,n}\left(x,y\right)\equiv\mathbf{K}^{\alpha,n}\left(\Psi\left(x\right),\Psi\left(y\right)\right)=\frac{\Psi\left(y\right)-\Psi\left(x\right)}{\left|\Psi\left(y\right)-\Psi\left(x\right)\right|^{n+1-\alpha}}, \qquad (4.2)$$

be the pullback of the kernel $\mathbf{K}^{\alpha,n}$ under Ψ, and define the corresponding operator

$$\left(\Psi^{*}\mathbf{R}^{\alpha,n}\right)\left(f\mu\right)\left(x\right)=\int\Psi^{*}\mathbf{K}^{\alpha,n}\left(x,y\right)f\left(y\right)d\mu\left(y\right).$$

We claim the equality

$$\mathfrak{N}_{\mathbf{R}^{\alpha,n}}\left(\sigma,\omega\right)=\mathfrak{N}_{\Psi^{*}\mathbf{R}^{\alpha,n}}\left(\Psi^{*}\sigma,\Psi^{*}\omega\right), \qquad (4.3)$$

where $\Psi^{*}\sigma=\left(\Psi^{-1}\right)_{*}\sigma$ denotes the pushforward of σ under Ψ^{-1}, but as Ψ is a homeomorphism we abuse notation by writing Ψ^{*} for $\left(\Psi^{-1}\right)_{*}$, and where $\mathfrak{N}_{\mathbf{R}^{\alpha,n}}\left(\sigma,\omega\right)$ is the best constant in the inequality

$$\int\left|\mathbf{R}^{\alpha,n}f\sigma\right|^{2}d\omega\leq\mathfrak{N}_{\mathbf{R}^{\alpha,n}}\left(\sigma,\omega\right)\int\left|f\right|^{2}d\sigma, \qquad (4.4)$$

and similarly for $\mathfrak{N}_{\Psi^{*}\mathbf{R}^{\alpha,n}}\left(\Psi^{*}\sigma,\Psi^{*}\omega\right)$. Indeed, with the change of variable $x'=\Psi\left(x\right)$, $y'=\Psi\left(y\right)$, and setting $\Psi^{*}f=f\circ\Psi$, etc., we have

$$\begin{aligned}
\int\left|\mathbf{R}^{\alpha,n}f\sigma\left(x'\right)\right|^{2}d\omega\left(x'\right)&=\int\left|\mathbf{R}^{\alpha,n}f\sigma\left(\Psi\left(x\right)\right)\right|^{2}d\Psi^{*}\omega\left(x\right);\\
\mathbf{R}^{\alpha,n}f\sigma\left(\Psi\left(x\right)\right)&=\int\mathbf{K}^{\alpha,n}\left(\Psi\left(x\right),y'\right)f\left(y'\right)d\sigma\left(y'\right)\\
&=\int\mathbf{K}^{\alpha,n}\left(\Psi\left(x\right),\Psi\left(y\right)\right)f\left(\Psi\left(y\right)\right)d\Psi^{*}\sigma\left(y\right)\\
&=\int\Psi^{*}\mathbf{K}^{\alpha,n}\left(x,y\right)\Psi^{*}f\left(y\right)d\Psi^{*}\sigma\left(y\right)\\
&=\left(\Psi^{*}\mathbf{R}^{\alpha,n}\right)\left(\Psi^{*}f\Psi^{*}\sigma\right)\left(x\right);\\
\int\left|f\left(y'\right)\right|^{2}d\sigma\left(y'\right)&=\int\left|\Psi^{*}f\left(y\right)\right|^{2}d\Psi^{*}\sigma\left(y\right),
\end{aligned}$$

which shows that (4.4) becomes

$$\int \left|\left(\Psi^* \mathbf{R}^{\alpha,n}\right)\left(\Psi^* f \Psi^* \sigma\right)(x)\right|^2 d\Psi^* \omega(x) \leq \mathfrak{N}_{\mathbf{R}^{\alpha,n}} \int \left|\Psi^* f(y)\right|^2 d\Psi^* \sigma(y),$$

and hence that (4.3) holds.

Now the operator $\Psi^* \mathbf{R}^{\alpha,n}$ is easily seen to be a standard fractional singular integral, but it fails to be a conformal Riesz transform in general because the phase $\Psi(y) - \Psi(x)$ in the kernel in (4.2) is not $y - x$. We will rectify this drawback by showing that the boundedness of the conformal Riesz transform $\mathbf{R}_{\Psi}^{\alpha,n}$ is equivalent to that of $\Psi^* \mathbf{R}^{\alpha,n}$, and that the appropriate testing conditions are equivalent as well. So consider the two inequalities (4.4) and

$$\int \left|\mathbf{R}_{\Psi}^{\alpha,n}\left(f \Psi^* \sigma\right)\right|^2 d\Psi^* \omega \leq \mathfrak{N}_{\mathbf{R}_{\Psi}^{\alpha,n}}\left(\left(\Psi^* \sigma, \Psi^* \omega\right)\right) \int |f|^2 d\Psi^* \sigma, \tag{4.5}$$

where we recall that the measures $\Psi^* \omega$ and $\Psi^* \sigma$ are the pushforwards under Ψ^{-1} of the measures ω and σ respectively. Here the constants $\mathfrak{N}_{\mathbf{R}^{\alpha,n}}(\sigma, \omega)$ and $\mathfrak{N}_{\mathbf{R}_{\Psi}^{\alpha,n}}\left(\Psi^* \sigma, \Psi^* \omega\right)$ are the smallest constants in their respective inequalities.

At this point we fix a collection of quasicubes $\Omega \mathcal{Q}^n$ with Ω biLipschitz, and recall the Muckenhoupt and energy constants

$$\begin{gathered}
\mathcal{A}_2^{\alpha}(\sigma, \omega),\ \mathcal{A}_2^{\alpha,\mathrm{dual}}(\sigma, \omega),\ A_2^{\alpha,\mathrm{punct}}(\sigma, \omega),\ A_2^{\alpha,\mathrm{punct,dual}}(\sigma, \omega),\ \mathcal{E}_{\alpha}^{\Omega \mathcal{Q}^n}(\sigma, \omega),\ \mathcal{E}_{\alpha}^{\Omega \mathcal{Q}^n,\mathrm{dual}}(\sigma, \omega); \\
\mathcal{A}_2^{\alpha}\left(\Psi^* \sigma, \Psi^* \omega\right),\ \mathcal{A}_2^{\alpha,\mathrm{dual}}\left(\Psi^* \sigma, \Psi^* \omega\right),\ A_2^{\alpha,\mathrm{punct}}\left(\Psi^* \sigma, \Psi^* \omega\right),\ A_2^{\alpha,\mathrm{punct,dual}}\left(\Psi^* \sigma, \Psi^* \omega\right), \\
\text{and } \mathcal{E}_{\alpha}^{\Omega \mathcal{Q}^n}\left(\Psi^* \sigma, \Psi^* \omega\right),\ \mathcal{E}_{\alpha}^{\Omega \mathcal{Q}^n,\mathrm{dual}}\left(\Psi^* \sigma, \Psi^* \omega\right),
\end{gathered}$$

that depend only on the measures and the quasicubes, and the testing constants

$$\begin{gathered}
\mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\Omega \mathcal{Q}^n}(\sigma, \omega),\ \mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\Omega \mathcal{Q}^n,\mathrm{dual}}(\sigma, \omega); \\
\mathfrak{T}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Omega \mathcal{Q}^n}\left(\Psi^* \sigma, \Psi^* \omega\right),\ \mathfrak{T}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Omega \mathcal{Q}^n,\mathrm{dual}}\left(\Psi^* \sigma, \Psi^* \omega\right),
\end{gathered}$$

that depend on the measures and the quasicubes as well as the fractional singular integral and its tangent line truncations. We sometimes suppress the dependence (σ, ω) on the measures when they are understood from the context, or when they do not play a significant role.

Finally, we define an even more general testing condition. Let $\mathcal{F}$ be a collection of bounded Borel sets, and let $\mathbf{T}^{\alpha,n}$ be an α-fractional singular integral. Then define $\mathfrak{T}_{\mathbf{T}^{\alpha,n}}^{\mathcal{F}} = \mathfrak{T}_{\mathbf{T}^{\alpha,n}}^{\mathcal{F}}(\sigma, \omega)$ to be the smallest constant in the inequality

$$\int \left|\mathbf{T}^{\alpha,n} \mathbf{1}_F \sigma\right|^2 d\omega \leq \mathfrak{T}_{\mathbf{T}^{\alpha,n}}^{\mathcal{F}} |F|_{\sigma}, \qquad F \in \mathcal{F},$$

and similarly for the dual $\mathfrak{T}_{\mathbf{T}^{\alpha,n,\mathrm{dual}}}^{\mathcal{F},\mathrm{dual}} = \mathfrak{T}_{\mathbf{T}^{\alpha,n,\mathrm{dual}}}^{\mathcal{F},\mathrm{dual}}(\sigma, \omega)$. Note that our testing conditions above are with $\mathcal{F} = \Omega \mathcal{Q}^n$ and $\mathbf{T}^{\alpha,n} = \mathbf{R}^{\alpha,n}$ or $\mathbf{R}_{\Psi}^{\alpha,n}$. Given Ψ and $\mathcal{F}$ as above, denote by $\Psi^* \mathcal{F} \equiv \left\{\Psi^{-1}(F) : F \in \mathcal{F}\right\}$ the pullback of $\mathcal{F}$ under the map Ψ, i.e. $\Psi^* \mathbf{1}_F = \mathbf{1}_F \circ \Psi = \mathbf{1}_{\Psi^{-1}(F)}$. Of particular interest for us is the set of quasicubes $\mathcal{Q} = \Omega \mathcal{Q}^n$ which is used in the versions given above of the testing conditions. Then we have $\Psi^* \mathcal{Q} = \left\{\Psi^{-1}(Q) : Q \in \mathcal{Q}\right\}$, and the sets $\Psi^{-1}(Q)$ form a new family of quasicubes since $\Psi^{-1} \circ \Omega$ is a globally biLipschitz map, which if necessary we will refer to as $\Psi^{-1} \circ \Omega$-quasicubes.

Our first proposition concerns the equivalence of the Muckenhoupt conditions under a biLipschitz change of variable, and the next proposition considers norm inequalities and testing conditions.

Proposition 28. *Suppose $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ is a globally biLipschitz map and that the Muckenhoupt conditions are defined by taking supremums over the collection $\Omega\mathcal{Q}^n$ of Ω-quasicubes. Let Ψ be another globally biLipschitz map, and let σ and ω be positive Borel measures possibly having common point masses. Then we have the following three equivalences:*

$$\begin{aligned}
A_2^{\alpha,\mathrm{punct}}(\sigma,\omega) &\approx A_2^{\alpha,\mathrm{punct}}(\Psi^*\sigma,\Psi^*\omega), \\
A_2^{\alpha,\mathrm{punct},\mathrm{dual}}(\sigma,\omega) &\approx A_2^{\alpha,\mathrm{punct},\mathrm{dual}}(\Psi^*\sigma,\Psi^*\omega), \\
\mathfrak{A}_2^{\alpha}(\sigma,\omega) &= \mathfrak{A}_2^{\alpha}(\Psi^*\sigma,\Psi^*\omega).
\end{aligned}$$

Note the absence of any statement regarding the one-tailed Muckenhoupt conditions with holes, $\mathcal{A}_2^{\alpha}$ and $\mathcal{A}_2^{\alpha,\mathrm{dual}}$, where it is not obvious that an equivalence is possible.

Proof. For convenience we set $\widetilde{\sigma} = \Psi^*\sigma$, $\widetilde{\omega} = \Psi^*\omega$ and $\widetilde{Q} = \Psi(Q)$. In order to show that

$$A_2^{\alpha,\mathrm{punct}}(\Psi^*\sigma,\Psi^*\omega) \lesssim A_2^{\alpha,\mathrm{punct}}(\sigma,\omega),$$

it suffices to show that

$$\frac{\widetilde{\omega}\left(K,\mathfrak{P}_{(\widetilde{\sigma},\widetilde{\omega})}\right)}{|K|^{1-\frac{\alpha}{n}}}\frac{|K|_{\widetilde{\sigma}}}{|K|^{1-\frac{\alpha}{n}}} \lesssim \sup_{Q\in\Omega\mathcal{Q}^n}\frac{\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)}{|Q|^{1-\frac{\alpha}{n}}}\frac{|Q|_{\sigma}}{|Q|^{1-\frac{\alpha}{n}}}$$

for all Ω-quasicubes $K \in \Omega\mathcal{Q}^n$. Now a change of variable shows that

$$\frac{\widetilde{\omega}\left(K,\mathfrak{P}_{(\widetilde{\sigma},\widetilde{\omega})}\right)}{|K|^{1-\frac{\alpha}{n}}}\frac{|K|_{\widetilde{\sigma}}}{|K|^{1-\frac{\alpha}{n}}} = \frac{\omega\left(\widetilde{K},\mathfrak{P}_{(\sigma,\omega)}\right)}{|K|^{1-\frac{\alpha}{n}}}\frac{\left|\widetilde{K}\right|_{\sigma}}{|K|^{1-\frac{\alpha}{n}}},$$

where of course $|K| \approx \left|\widetilde{K}\right|$. Now choose a quasicube Q containing $\widetilde{K}$ with $\ell(Q) \leq C_{\Omega}\ell(K)$ so that we have $|K| \approx |Q|$. If a largest common point mass for ω in $\widetilde{K}$ (respectively Q) occurs at x (respectively y), then $\omega(\{x\}) \leq \omega(\{y\})$ and so we have

$$\omega\left(\widetilde{K},\mathfrak{P}_{(\sigma,\omega)}\right) = \left|\widetilde{K}\right|_{\omega} - \omega(\{x\})\,\delta_x \leq |Q|_{\omega} - \omega(\{y\})\,\delta_y = \omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right),$$

since if $x = y$ we use $\left|\widetilde{K}\right|_{\omega} \leq |Q|_{\omega}$, while if $x \neq y$, then $y \notin \widetilde{K}$ and we use $|Q|_{\omega} - \omega(\{y\})\,\delta_y \geq \left|\widetilde{K}\right|_{\omega}$. Thus

$$\begin{aligned}
\frac{\widetilde{\omega}\left(K,\mathfrak{P}_{(\widetilde{\sigma},\widetilde{\omega})}\right)}{|K|^{1-\frac{\alpha}{n}}}\frac{|K|_{\widetilde{\sigma}}}{|K|^{1-\frac{\alpha}{n}}} &\leq \frac{\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)}{|K|^{1-\frac{\alpha}{n}}}\frac{\left|\widetilde{K}\right|_{\sigma}}{|K|^{1-\frac{\alpha}{n}}} \\
&\lesssim \frac{\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)}{|Q|^{1-\frac{\alpha}{n}}}\frac{|Q|_{\sigma}}{|Q|^{1-\frac{\alpha}{n}}} \leq A_2^{\alpha,\mathrm{punct}}(\sigma,\omega),
\end{aligned}$$

which completes the proof of the first assertion in Proposition 28. The second assertion is proved in similar fashion.

Now we turn to the third assertion in Proposition 28, where in view of what we have just shown, it suffices to show that

$$\mathcal{A}_2^{\alpha}(\widetilde{\sigma},\widetilde{\omega}) + \mathcal{A}_2^{\alpha,\mathrm{dual}}(\widetilde{\sigma},\widetilde{\omega}) \lesssim \mathfrak{A}_2^{\alpha}(\sigma,\omega).$$

By symmetry it is then enough to show

$$\mathcal{P}\left(K,\mathbf{1}_{K^c}\widetilde{\sigma}\right)\frac{|K|_{\widetilde{\omega}}}{|K|^{1-\frac{\alpha}{n}}}\lesssim\mathfrak{A}_2^{\alpha}\left(\sigma,\omega\right),$$

for all Ω-quasicubes $K\in\Omega\mathcal{Q}^n$. Now a change of variable shows that

$$\begin{aligned}\frac{|K|_{\widetilde{\omega}}}{|K|^{1-\frac{\alpha}{n}}}\mathcal{P}\left(K,\mathbf{1}_{K^c}\widetilde{\sigma}\right) &= \frac{|K|_{\widetilde{\omega}}}{|K|^{1-\frac{\alpha}{n}}}\int_{\mathbb{R}^n\setminus K}\left(\frac{|K|^{\frac{1}{n}}}{\left(|K|^{\frac{1}{n}}+|x-c_K|\right)^2}\right)^{n-\alpha}d\widetilde{\sigma}\left(x\right)\\ &\approx \frac{\left|\widetilde{K}\right|_{\omega}}{|K|^{1-\frac{\alpha}{n}}}\int_{\mathbb{R}^n\setminus\widetilde{K}}\left(\frac{|K|^{\frac{1}{n}}}{\left(|K|^{\frac{1}{n}}+\left|x'-c_{\widetilde{K}}\right|\right)^2}\right)^{n-\alpha}d\sigma\left(x'\right)\\ &\approx \frac{\left|\widetilde{K}\right|_{\omega}}{|K|^{1-\frac{\alpha}{n}}}\mathcal{P}\left(\widetilde{K},\mathbf{1}_{\widetilde{K}^c}\sigma\right)\approx\frac{\left|\widetilde{K}\right|_{\omega}}{|K|^{1-\frac{\alpha}{n}}}\mathcal{P}\left(K,\mathbf{1}_{\widetilde{K}^c}\sigma\right),\end{aligned}$$

where of course $|K|\approx\left|\widetilde{K}\right|$ and $\mathcal{P}\left(K,\mu\right)\approx\mathcal{P}\left(\widetilde{K},\mu\right)$. We have written $\widetilde{K}$ in place of K in the final equivalence only when it matters. Now choose quasicubes $Q,P\in\Omega\mathcal{Q}^n$ such that

$$Q\subset\widetilde{K}\subset P\text{ and }\ell\left(P\right)\leq C_{\Omega}\ell\left(Q\right),$$

so that $|Q|\approx\left|\widetilde{K}\right|\approx|P|$, and $\mathcal{P}\left(Q,\mu\right)\approx\mathcal{P}\left(\widetilde{K},\mu\right)\approx\mathcal{P}\left(P,\mu\right)$ for any postive measure μ. Let $y\in P$ be a point where the largest common point mass of σ occurs, and let $z\in P$ be a point where the largest common point mass of ω occurs. Define

$$\dot{\sigma}=\sigma-\sigma\left(\{y\}\right)\text{ and }\dot{\omega}=\omega-\omega\left(\{z\}\right).$$

Now we have the two 'punctured' inequalities,

$$\begin{aligned}\frac{\left|\widetilde{K}\right|_{\omega}}{|K|^{1-\frac{\alpha}{n}}}\mathcal{P}\left(K,\mathbf{1}_{\widetilde{K}^c}\dot{\sigma}\right) &\leq \frac{|P|_{\omega}}{|K|^{1-\frac{\alpha}{n}}}\mathcal{P}\left(K,\mathbf{1}_{P}\dot{\sigma}\right)+\frac{|P|_{\omega}}{|K|^{1-\frac{\alpha}{n}}}\mathcal{P}\left(K,\mathbf{1}_{P^c}\sigma\right)\\ &\lesssim A_2^{\alpha,\text{punct}}\left(\sigma,\omega\right)+\mathcal{A}_2^{\alpha}\left(\sigma,\omega\right)\leq\mathfrak{A}_2^{\alpha}\left(\sigma,\omega\right),\end{aligned}\tag{4.6}$$

and

$$\begin{aligned}\frac{\left|\widetilde{K}\right|_{\dot{\omega}}}{|K|^{1-\frac{\alpha}{n}}}\mathcal{P}\left(K,\mathbf{1}_{\widetilde{K}^c}\sigma\right) &\leq \frac{|P|_{\dot{\omega}}}{|K|^{1-\frac{\alpha}{n}}}\mathcal{P}\left(K,\mathbf{1}_{P}\sigma\right)+\frac{|P|_{\omega}}{|K|^{1-\frac{\alpha}{n}}}\mathcal{P}\left(K,\mathbf{1}_{P^c}\sigma\right)\\ &\lesssim A_2^{\alpha,\text{punct,dual}}\left(\sigma,\omega\right)+\mathcal{A}_2^{\alpha}\left(\sigma,\omega\right)\leq\mathfrak{A}_2^{\alpha}\left(\sigma,\omega\right).\end{aligned}\tag{4.7}$$

Next, we claim that if $y\neq z$, then

$$\frac{\sigma\left(\{y\}\right)}{|P|^{1-\frac{\alpha}{n}}}\frac{\omega\left(\{z\}\right)}{|P|^{1-\frac{\alpha}{n}}}\lesssim\mathcal{A}_2^{\alpha}\left(\sigma,\omega\right)+\mathcal{A}_2^{\alpha,\text{dual}}\left(\sigma,\omega\right)\leq\mathfrak{A}_2^{\alpha}\left(\sigma,\omega\right).\tag{4.8}$$

Indeed, it is easy to find a quasicube $R\subset P$ with half the side length of P such that exactly one of y and z lies in R. For the purpose of clarifying the remainder of this argument, we assume these quasicubes are all ordinary cubes, and the reader can then easily modify the argument to quasicubes. If y and z lie on opposite sides of a horizontal or vertical line L, then $P\setminus L$ consists of two disjoint rectangles,

each containing one of the points y and z. Clearly, the larger of the two rectangles contains a cube R with side length at least $\frac{1}{2}\ell(P)$ and containing one of y and z.

With such a quasicube R in hand, say with $y \in R$ and $z \in R^c$, then

$$\frac{\sigma(\{y\})}{|P|^{1-\frac{\alpha}{n}}}\frac{\omega(\{z\})}{|P|^{1-\frac{\alpha}{n}}} \lesssim \frac{|R|_\sigma}{|R|^{1-\frac{\alpha}{n}}}\mathcal{P}(R, \mathbf{1}_{R^c}\omega) \leq \mathcal{A}_2^{\alpha,\mathrm{dual}}(\sigma,\omega) \leq \mathfrak{A}_2^\alpha(\sigma,\omega).$$

Now we write

$$\begin{aligned}
\frac{\left|\widetilde{K}\right|_\omega}{|K|^{1-\frac{\alpha}{n}}}\mathcal{P}\left(K, \mathbf{1}_{\widetilde{K}^c}\sigma\right) &= \frac{\left|\widetilde{K}\right|_{\dot{\omega}+\omega(\{z\})}}{|K|^{1-\frac{\alpha}{n}}}\mathcal{P}\left(K, \mathbf{1}_{\widetilde{K}^c}\left[\dot{\sigma}+\sigma(\{y\})\right]\right) \\
&\leq \frac{\left|\widetilde{K}\right|_{\dot{\omega}}}{|K|^{1-\frac{\alpha}{n}}}\mathcal{P}\left(K, \mathbf{1}_{\widetilde{K}^c}\sigma\right) + \frac{\left|\widetilde{K}\right|_{\omega}}{|K|^{1-\frac{\alpha}{n}}}\mathcal{P}\left(K, \mathbf{1}_{\widetilde{K}^c}\dot{\sigma}\right) + \frac{\sigma(\{y\})}{|P|^{1-\frac{\alpha}{n}}}\frac{\omega(\{z\})}{|P|^{1-\frac{\alpha}{n}}}.
\end{aligned}$$

If $y \neq z$ then we have

$$\frac{\left|\widetilde{K}\right|_\omega}{|K|^{1-\frac{\alpha}{n}}}\mathcal{P}\left(K, \mathbf{1}_{\widetilde{K}^c}\sigma\right) \lesssim \mathfrak{A}_2^\alpha(\sigma,\omega)$$

by the three inequalities (4.6), (4.7) and (4.8) proved above. On the other hand, if $y = z$, then either $y \in \widetilde{K}$ or $z \in P \setminus \widetilde{K}$. If $y \in \widetilde{K}$ we have

$$\frac{\left|\widetilde{K}\right|_\omega}{|K|^{1-\frac{\alpha}{n}}}\mathcal{P}\left(K, \mathbf{1}_{\widetilde{K}^c}\sigma\right) \leq \frac{\left|\widetilde{K}\right|_\omega}{|K|^{1-\frac{\alpha}{n}}}\mathcal{P}\left(K, \mathbf{1}_{\widetilde{K}^c}\dot{\sigma}\right) \lesssim \mathfrak{A}_2^\alpha(\sigma,\omega)$$

by (4.6), and if $z \in P \setminus \widetilde{K}$ we have

$$\frac{\left|\widetilde{K}\right|_\omega}{|K|^{1-\frac{\alpha}{n}}}\mathcal{P}\left(K, \mathbf{1}_{\widetilde{K}^c}\sigma\right) \leq \frac{\left|\widetilde{K}\right|_{\dot{\omega}}}{|K|^{1-\frac{\alpha}{n}}}\mathcal{P}\left(K, \mathbf{1}_{\widetilde{K}^c}\sigma\right) \lesssim \mathfrak{A}_2^\alpha(\sigma,\omega)$$

by (4.7). This completes the proof of Proposition 28. □

Proposition 29. *Suppose $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is a $C^{1,\delta}$ diffeomorphism, i.e. both Ψ and its inverse Ψ^{-1} are globally $C^{1,\delta}$ maps, let σ and ω be positive Borel measures (possibly having common point masses) with one of the measures supported in a* compact *subset K of $\mathbb{R}^n$, and let $\mathcal{F}$ be a collection of bounded Borel sets. Then with the fractional Riesz transform $\mathbf{R}^{\alpha,n}$ and the conformal fractional Riesz transform $\mathbf{R}_\Psi^{\alpha,n}$ as above, we have the following three equivalences:*

1. $\mathfrak{N}_{\mathbf{R}^{\alpha,n}}(\sigma,\omega) + \sqrt{\mathfrak{A}_2^\alpha(\sigma,\omega)} \approx \mathfrak{N}_{\mathbf{R}_\Psi^{\alpha,n}}(\Psi^*\sigma,\Psi^*\omega) + \sqrt{\mathfrak{A}_2^\alpha(\sigma,\omega)}$,
2. $\mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\mathcal{F}}(\sigma,\omega) + \sqrt{\mathfrak{A}_2^\alpha(\sigma,\omega)} \approx \mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Psi^*\mathcal{F}}(\Psi^*\sigma,\Psi^*\omega) + \sqrt{\mathfrak{A}_2^\alpha(\sigma,\omega)}$,
3. $\mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\mathcal{F},\mathrm{dual}}(\sigma,\omega) + \sqrt{\mathfrak{A}_2^\alpha(\sigma,\omega)} \approx \mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Psi^*\mathcal{F},\mathrm{dual}}(\Psi^*\sigma,\Psi^*\omega) + \sqrt{\mathfrak{A}_2^\alpha(\sigma,\omega)}$,

where the implied constants depend only $n, \alpha, \mathrm{diam}(K), \|\Gamma\|_{1,\delta}$ and $\|\Psi\|_{C^{1,\delta}}+\left\|\Psi^{-1}\right\|_{C^{1,\delta}}$.

In particular, we see that in the presence of the Muckenhoupt conditions $\mathfrak{A}_2^\alpha$, and when one of the measures is supported in a compact set, the testing conditions for $\mathbf{T}_\Gamma^\alpha$ on indicators of quasicubes Q are equivalent to the testing conditions for $\mathbf{T}_{\Gamma_\Psi}^{\alpha,n}$ on indicators of the new quasicubes $\Psi^{-1}(Q)$. Thus in order to deform the sets over which we test from Q to $\Psi^{-1}(Q)$, we need only push the measures forward and

alter the conformal factor Γ to the associated conformal factor Γ_Ψ in the operator, keeping the critical phase $y-x$ in the numerator of the kernel unchanged. The same results hold for the inverse $C^{1,\delta}$ diffeomorphism Ψ^{-1} in place of Ψ.

Before beginning the proof it is convenient to introduce two auxilliary operators $\Psi^{*,\tan,1}\mathbf{R}^{\alpha,n}$ and $\Psi^{*,\tan,2}\mathbf{R}^{\alpha,n}$ with kernels related to the pullback kernel $\Psi^*\mathbf{K}^{\alpha,n}$ defined above by

$$\Psi^*\mathbf{K}^{\alpha,n}(x,y)=\frac{\Psi(y)-\Psi(x)}{|\Psi(y)-\Psi(x)|^{n+1-\alpha}}.$$

We define the kernels of $\Psi^{*,\tan,1}\mathbf{R}^{\alpha,n}$ and $\Psi^{*,\tan,2}\mathbf{R}^{\alpha,n}$ by

$$\begin{aligned}\Psi^{*,\tan,1}\mathbf{K}^{\alpha,n}(x,y) &\equiv \Psi'(x)\ \frac{y-x}{|\Psi(y)-\Psi(x)|^{n+1-\alpha}},\\ \Psi^{*,\tan,2}\mathbf{K}^{\alpha,n}(x,y) &\equiv \Psi'(y)\ \frac{y-x}{|\Psi(y)-\Psi(x)|^{n+1-\alpha}}.\end{aligned}$$

The superscript $\tan,1$ (respectively $\tan,2$) indicates that we are replacing the phase function $\Psi(y)-\Psi(x)$ with its tangent line approximation at x (respectively y). Now we prove Proposition 29.

Proof. We begin with the first statement, where we may assume that ω is supported in a compact ball B. Moreover, we may assume the cubes below are usual cubes since we invoke no testing or energy in the proof of this second statement. Then we may also assume that σ is supported in the double $2B$. Indeed, if ω is supported in a ball B and σ is supported outside the double $2B$ of the ball B, then the associated norm inequality for a fractional singular integral operator $\mathbf{T}^{\alpha,n}$ is easily seen to be controlled solely in terms of the Muckenhoupt constant $\mathcal{A}_2^\alpha$ with holes:

(4.9)
$$\begin{aligned}\int_{\mathbb{R}^n\setminus 2B}\left|\mathbf{T}^{\alpha,n}\mathbf{1}_B g\omega\right|^2 d\sigma &\lesssim \int_{\mathbb{R}^n\setminus 2B}\left|\int_B |x-y|^{\alpha-n} g(x)\,d\omega(x)\right|^2 d\sigma(y)\\ &\lesssim \left(\int_B |g|^2 d\omega\right)\int_{\mathbb{R}^n\setminus 2B}\left(\int_B |x-y|^{2\alpha-2n}d\omega(x)\right)d\sigma(y)\\ &\lesssim \|g\|^2_{L^2(\omega)}\frac{|B|_\omega}{|B|^{1-\frac{\alpha}{n}}}\mathcal{P}^\alpha(B,\sigma)\lesssim \mathcal{A}_2^\alpha\|g\|^2_{L^2(\omega)}\ ,\end{aligned}$$

where we have used that $|x-y|^{2\alpha-2n}\approx |c_B-y|^{2\alpha-2n}$ for $y\in\mathbb{R}^n\setminus 2B$ and $x\in B$.

We write the pullback $\Psi^*\mathbf{K}^{\alpha,n}(x,y)$ of the vector kernel $\mathbf{K}^{\alpha,n}(x,y)=\frac{y-x}{|y-x|^{n+1-\alpha}}$, given in formula (4.2), in the form

$$\begin{aligned}\Psi^*\mathbf{K}^{\alpha,n}(x,y) &= \frac{\Psi(y)-\Psi(x)}{|\Psi(y)-\Psi(x)|^{n+1-\alpha}}\\ &\equiv \frac{\Psi'(x)(y-x)}{|\Psi(y)-\Psi(x)|^{n+1-\alpha}}+\mathbf{E}_1^\alpha(x,y)\\ &= \Psi^{*,\tan,1}\mathbf{K}^{\alpha,n}(x,y)+\mathbf{E}_1^\alpha(x,y)\,,\end{aligned}\tag{4.10}$$

where we write Ψ' for the derivative $D\Psi$, and where $\Psi^{*,\tan,1}\mathbf{K}^{\alpha,n}$ is the first of our auxilliary kernels defined above. We claim that the error kernel $\mathbf{E}_1^\alpha(x,y)$ satisfies the improved local estimate

$$\mathbf{E}_1^\alpha(x,y)=O\left(|y-x|^{\alpha-n+\delta};M\right)\tag{4.11}$$

for some constant M depending on $\|\Gamma\|_{1,\delta}$, $\|\Psi\|_{C^{1,\delta}}$ and $\left\|(\Psi')^{-1}\right\|_{\infty}$. Indeed, we have

$$\begin{aligned} & \frac{\Psi(y)-\Psi(x)}{|\Psi(y)-\Psi(x)|^{n+1-\alpha}}-\frac{\Psi'(x)(y-x)}{|\Psi(y)-\Psi(x)|^{n+1-\alpha}} \\ = \quad & \frac{1}{|\Psi(y)-\Psi(x)|^{n+1-\alpha}}\left\{\Psi(y)-\Psi(x)-\Psi'(x)(y-x)\right\}. \end{aligned}$$

Now we use the estimate
(4.12)

$$|\Psi(y)-\Psi(x)-\Psi'(x)(y-x)| \leq \|\Psi\|_{C^{1+\delta}}|y-x|^{1+\delta}=O\left(|y-x|^{1+\delta};\|\Psi\|_{C^{1+\delta}}\right),$$

together with the bound $|\Gamma(\Psi(x),\Psi(y))| \leq \|\Gamma\|_{\infty}$, to obtain (4.11):

$$|\mathbf{E}_1^{\alpha}(x,y)| \lesssim \frac{1}{|y-x|^{n+1-\alpha}}|y-x|^{1+\delta}=|y-x|^{\alpha-n+\delta},$$

for $|y-x| \leq C$.

Recall that both $\widetilde{\sigma}$ and $\widetilde{\omega}$ are supported in a fixed compact set K. Let C_{2B} be a sufficiently large constant exceeding the diameter of $2B$. Now we bound the operator norm of the error term. For this we write $\widetilde{\sigma}=\Psi^*\sigma$ and $\widetilde{\omega}=\Psi^*\omega$ for convenience, and then observe that the norm of the error operator

$$\mathcal{E}_1^{\alpha} f\widetilde{\sigma}(x) \equiv \int \mathbf{E}_1^{\alpha}(x,y) f(y)\, d\widetilde{\sigma}(y)$$

as a map from $L^2(\widetilde{\sigma})$ to $L^2(\widetilde{\omega})$ is controlled by the offset A_2^{α} constant. Indeed, from the definition of $\mathbf{E}_1^{\alpha}(x,y)$ in (4.10), we see that the kernel $\mathbf{E}_1^{\alpha}$ vanishes on the diagonal, and so for $a \sim \log_2 \frac{1}{C_K}$,

$$\begin{aligned} |\mathcal{E}_1^{\alpha} f\widetilde{\sigma}(x)| &\leq \sum_{k=a}^{\infty} \int_{B(x,2^{-k})\setminus B(x,2^{-k-1})} M|y-x|^{\alpha+\delta-n}|f(y)|\, d\widetilde{\sigma}(y) \\ &\lesssim \sum_{k=a}^{\infty} 2^{-k\delta}\left|B\left(x,2^{-k}\right)\right|^{\frac{\alpha}{n}-1} \int_{B(x,2^{-k})\setminus B(x,2^{-k-1})} |f(y)|\, d\widetilde{\sigma}(y) \\ &\lesssim \sum_{k=a}^{\infty} 2^{-k\delta}\mathcal{A}_{\alpha}^{k}(f\widetilde{\sigma})(x), \end{aligned}$$

where $\mathcal{A}_{\alpha,C_K}^{k}$ is the *annular* α-averaging operator given by

$$\mathcal{A}_{\alpha}^{k}(f\widetilde{\sigma})(x) \equiv \left|B\left(x,2^{-k}\right)\right|^{\frac{\alpha}{n}-1}\int_{B(x,2^{-k})\setminus B(x,2^{-k-1})} |f|\, d\widetilde{\sigma}.$$

We now claim that the boundedness of $\mathcal{A}_{\alpha}^{k}$, and hence also that of $\mathcal{E}_1^{\alpha}$, is controlled by the offset A_2^{α} constant. Indeed, for a sufficiently small positive constant

c, we have

$$\begin{aligned}
\left\|\mathcal{A}_{\alpha}^{k}\left(f\widetilde{\sigma}\right)\right\|_{L^{2}(\widetilde{\omega})}^{2} &\leq \int_{\mathbb{R}^{n}}\left(2^{-k(\alpha-n)}\int_{B(x,2^{-k})\setminus B(x,2^{-k-1})}|f|\,d\widetilde{\sigma}\right)^{2}d\widetilde{\omega}\left(x\right)\\
&\leq 2^{-2k(\alpha-n)}\int_{\mathbb{R}^{n}}\left(\int_{B(x,2^{-k})}|f|^{2}\,d\widetilde{\sigma}\right)\left|B\left(x,2^{-k}\right)\setminus B\left(x,2^{-k-1}\right)\right|_{\widetilde{\sigma}}d\widetilde{\omega}\left(x\right)\\
&= 2^{-2k(\alpha-n)}\sum_{z\in\mathbb{Z}^{n}}\int_{B\left(c2^{-k}z,\sqrt{n}c2^{-k}\right)}\left(\int_{B(x,2^{-k})}|f|^{2}\,d\widetilde{\sigma}\right)\left|B\left(x,2^{-k}\right)\setminus B\left(x,2^{-k-1}\right)\right|_{\widetilde{\sigma}}d\widetilde{\omega}\left(x\right)\\
&\leq 2^{-2k(\alpha-n)}\sum_{z\in\mathbb{Z}^{n}}\int_{B\left(c2^{-k}z,\sqrt{n}c2^{-k}\right)}\left(\int_{B\left(c2^{-k}z,\left(\sqrt{n}c+1\right)2^{-k}\right)}|f|^{2}\,d\widetilde{\sigma}\right)\\
&\qquad\qquad\times\left|B\left(c2^{-k}z,\left(\sqrt{n}c+1\right)2^{-k}\right)\setminus B\left(c2^{-k}z,nc2^{-k}\right)\right|_{\widetilde{\sigma}}d\widetilde{\omega}\left(x\right),\\
&= \sum_{z\in\mathbb{Z}^{n}}\frac{\left|B\left(c2^{-k}z,\sqrt{n}c2^{-k}\right)\right|_{\widetilde{\omega}}\left|B\left(c2^{-k}z,\left(\sqrt{n}c+1\right)2^{-k}\right)\setminus B\left(cn2^{-k}z,nc2^{-k}\right)\right|_{\widetilde{\sigma}}}{2^{2k(\alpha-n)}}\\
&\qquad\qquad\times\int_{B\left(2^{-k}z,\left(\sqrt{n}+1\right)2^{-k}\right)}|f|^{2}\,d\widetilde{\sigma}\ .
\end{aligned}$$

Using the separation between $B\left(c2^{-k}z,\left(\sqrt{n}c+1\right)2^{-k}\right)\setminus B\left(cn2^{-k}z,nc2^{-k}\right)$ and $B\left(c2^{-k}z,\sqrt{n}c2^{-k}\right)$, it is easy to see that

$$\frac{\left|B\left(c2^{-k}z,\sqrt{n}c2^{-k}\right)\right|_{\widetilde{\omega}}\ \left|B\left(c2^{-k}z,\left(\sqrt{n}c+1\right)2^{-k}\right)\setminus B\left(cn2^{-k}z,nc2^{-k}\right)\right|_{\widetilde{\sigma}}}{2^{2k(\alpha-n)}}\lesssim A_{2}^{\alpha}\ .$$

Combining inequalities we then obtain

$$\begin{aligned}
\left\|\mathcal{A}_{\alpha}^{k}\left(f\widetilde{\sigma}\right)\right\|_{L^{2}(\widetilde{\omega})}^{2} &\lesssim \sum_{z\in\mathbb{Z}^{n}}A_{2}^{\alpha}\int_{B\left(2^{-k}z,\left(\sqrt{n}+1\right)2^{-k}\right)}|f|^{2}\,d\widetilde{\sigma}\\
&= A_{2}^{\alpha}\int_{\mathbb{R}^{n}}\sum_{z\in\mathbb{Z}^{n}}\mathbf{1}_{B\left(2^{-k}z,\left(\sqrt{n}+1\right)2^{-k}\right)}|f|^{2}\,d\widetilde{\sigma}\\
&\lesssim A_{2}^{\alpha}\int_{\mathbb{R}^{n}}|f|^{2}\,d\widetilde{\sigma}=A_{2}^{\alpha}\left\|f\right\|_{L^{2}(\widetilde{\sigma})}^{2}\ ,
\end{aligned}$$

and hence

$$\left\|\mathcal{E}_{1}^{\alpha}f\widetilde{\sigma}\right\|_{L^{2}(\widetilde{\omega})}\lesssim\sum_{k=a}^{\infty}2^{-k\delta}\left\|\mathcal{A}_{\alpha}^{k}\left(f\widetilde{\sigma}\right)\right\|_{L^{2}(\widetilde{\omega})}\lesssim\sum_{k=a}^{\infty}2^{-k\delta}A_{2}^{\alpha}\left\|f\right\|_{L^{2}(\widetilde{\sigma})}\lesssim C_{2B}^{\delta}A_{2}^{\alpha}\left\|f\right\|_{L^{2}(\widetilde{\sigma})}\ .$$

This completes the proof that the norm of the error operator $\mathcal{E}_{1}^{\alpha}$ as a map from $L^{2}\left(\widetilde{\sigma}\right)$ to $L^{2}\left(\widetilde{\omega}\right)$ is controlled by the offset A_{2}^{α} constant. For reference in proving statements (2) and (3) below, we record that in similar fashion, using the reduction that $\widetilde{\sigma}$ also has compact support, that the norm of the dual error operator

$$\begin{aligned}
\mathcal{E}_{2}^{\alpha}f\widetilde{\sigma}\left(x\right) &\equiv \int\mathbf{E}_{2}^{\alpha}\left(x,y\right)f\left(y\right)d\widetilde{\sigma}\left(y\right);\\
\Psi^{*}\mathbf{K}^{\alpha,n}\left(x,y\right) &= \Psi^{*,\tan,2}\mathbf{K}^{\alpha,n}\left(x,y\right)+\mathbf{E}_{2}^{\alpha}\left(x,y\right),
\end{aligned}$$

as a map from $L^{2}\left(\widetilde{\omega}\right)$ to $L^{2}\left(\widetilde{\sigma}\right)$ is controlled by the offset A_{2}^{α} constant.

Now we further analyze the first kernel on the right hand side of (4.10), namely

$$\Psi^{*,\tan,1}\mathbf{K}^{\alpha,n}(x,y)=\frac{\Psi'(x)(y-x)}{|\Psi(y)-\Psi(x)|^{n+1-\alpha}},$$

by writing

$$\Psi^{*,\tan,1}\mathbf{K}^{\alpha,n}(x,y)=\Psi'(x)\frac{y-x}{|\Psi(y)-\Psi(x)|^{n+1-\alpha}}=\Psi'(x)\mathbf{K}_{\Psi}^{\alpha,n}(x,y).$$

We now compute that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left|\Psi^{*,\tan,1}\mathbf{R}^{\alpha,n}f\widetilde{\sigma}\right|^2 d\widetilde{\omega} \qquad (4.13)\\
=&\int_{\mathbb{R}^n}\left|\int\Psi^{*,\tan,1}\mathbf{K}^{\alpha,n}(x,y)f(y)\,d\widetilde{\sigma}(y)\right|^2 d\widetilde{\omega}(x)\\
=&\int_{\mathbb{R}^n}\left|\int\Psi'(x)\mathbf{K}_{\Psi}^{\alpha,n}(x,y)f(y)\,d\widetilde{\sigma}(y)\right|^2 d\widetilde{\omega}(x)\\
=&\int_{\mathbb{R}^n}\left|\Psi'(x)\left\{\int\mathbf{K}_{\Psi}^{\alpha,n}(x,y)f(y)\,d\widetilde{\sigma}(y)\right\}\right|^2 d\widetilde{\omega}(x)\\
=&\int_{\mathbb{R}^n}\left|\Psi'(x)\int\mathbf{R}_{\Psi}^{\alpha,n}(f\widetilde{\sigma})(x)\right|^2 d\widetilde{\omega}(x),
\end{aligned}$$

where the matrix $\Psi'(x)$ is acting on the vector $\mathbf{T}_{\Gamma_\Psi}^{\alpha,n}f\widetilde{\sigma}(x)=\int\mathbf{K}_{\Gamma_\Psi}^{\alpha,n}(x,y)f(y)\,d\widetilde{\sigma}(y)$. Using the inequality

$$|\Psi'(x)v|\approx|v|,\qquad\text{uniformly in }x, \tag{4.14}$$

we conclude that

$$\int_{\mathbb{R}^n}\left|\Psi^{*,\tan,1}\mathbf{R}^{\alpha,n}f\widetilde{\sigma}\right|^2 d\widetilde{\omega}\approx\int_{\mathbb{R}^n}\left|\mathbf{R}_{\Psi}^{\alpha,n}f\widetilde{\sigma}\right|^2 d\widetilde{\omega},$$

which shows that

$$\mathfrak{N}_{\mathbf{R}_{\Psi}^{\alpha,n}}(\widetilde{\sigma},\widetilde{\omega})\approx\mathfrak{N}_{\Psi^{*,\tan,1}\mathbf{R}^{\alpha,n}}(\widetilde{\sigma},\widetilde{\omega}).$$

Similarly we have

$$\mathfrak{N}_{\mathbf{R}_{\Psi}^{\alpha,n}}(\widetilde{\sigma},\widetilde{\omega})\approx\mathfrak{N}_{\Psi^{*,\tan,2}\mathbf{R}^{\alpha,n}}(\widetilde{\sigma},\widetilde{\omega}).$$

Reverting to the notation with Ψ^* it now follows from this, and then the boundedness of the error operator $\mathcal{E}^\alpha$, that

$$\begin{aligned}
\mathfrak{N}_{\mathbf{R}_{\Psi}^{\alpha}}(\Psi^*\sigma,\Psi^*\omega)&\approx\mathfrak{N}_{\Psi^{*,\tan,1}\mathbf{R}^{\alpha,n}}(\Psi^*\sigma,\Psi^*\omega)\\
&\lesssim\mathfrak{N}_{\Psi^*\mathbf{R}^{\alpha,n}}(\Psi^*\sigma,\Psi^*\omega)+\left[\mathcal{A}_2^\alpha+\mathcal{A}_2^{\alpha,\mathrm{dual}}\right]\\
&=\mathfrak{N}_{\mathbf{R}^\alpha}(\sigma,\omega)+\left[\mathcal{A}_2^\alpha+\mathcal{A}_2^{\alpha,\mathrm{dual}}\right],
\end{aligned}$$

where the final equality is (4.3). The reverse inequality in the second statement of Proposition 29 is proved in similar fashion, or simply by replacing Ψ with Ψ^{-1}.

The second and third statements are proved in the same way as the first statement just proved above, but with the following difference. The functions f under consideration are restricted to indicators $f=\mathbf{1}_E$ with $E\in\Psi^*\mathcal{F}$, and as a result we have from (4.13), and its dual version, the two indentities

$$\int_{\mathbb{R}^n}\left|\Psi^{*,\tan,1}\mathbf{R}^{\alpha,n}\mathbf{1}_E\widetilde{\sigma}\right|^2 d\widetilde{\omega}=\int_{\mathbb{R}^n}\left|\Psi'(x)\mathbf{R}_{\Psi}^{\alpha,n}(\mathbf{1}_E\widetilde{\sigma})(x)\right|^2 d\widetilde{\omega}(x),$$

and

$$\int_{\mathbb{R}^n} \left| \Psi^{*,\tan,2} \mathbf{R}^{\alpha,n} \mathbf{1}_E \widetilde{\omega} \right|^2 d\widetilde{\sigma} = \int_{\mathbb{R}^n} \left| \Psi'(y) \mathbf{R}^{\alpha,n}_{\Psi} \left(\mathbf{1}_E \widetilde{\omega} \right)(y) \right|^2 d\widetilde{\sigma}(y),$$

since the kernels $\mathbf{K}^{\alpha,n}(x,y)$ and $\mathbf{K}^{\alpha,n}_{\Gamma_\Psi}(x,y)$ are antisymmetric. Just as for the norm estimate above, we use (4.14) to obtain both

$$\begin{aligned} \int_{\mathbb{R}^n} \left| \Psi^{*,\tan,1} \mathbf{R}^{\alpha,n} \mathbf{1}_E \widetilde{\sigma} \right|^2 d\widetilde{\omega} &\approx \int_{\mathbb{R}^n} \left| \mathbf{R}^{\alpha,n}_{\Psi}(x,y) \mathbf{1}_E d\widetilde{\sigma} \right|^2 d\widetilde{\omega}; \\ \mathfrak{T}^{\Psi^*\mathcal{F}}_{\Psi^{*,\tan,1}\mathbf{R}^{\alpha,n}} \left(\Psi^*\sigma, \Psi^*\omega \right) &\approx \mathfrak{T}^{\Psi^*\mathcal{F}}_{\mathbf{R}^{\alpha}_{\Psi}} \left(\Psi^*\sigma, \Psi^*\omega \right), \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^n} \left| \Psi^{*,\tan,2} \mathbf{R}^{\alpha,n} \mathbf{1}_E \widetilde{\omega} \right|^2 d\widetilde{\sigma} &\approx \int_{\mathbb{R}^n} \left| \mathbf{R}^{\alpha,n}_{\Psi}(x,y) \mathbf{1}_E d\widetilde{\omega} \right|^2 d\widetilde{\sigma}; \\ \mathfrak{T}^{\Psi^*\mathcal{F},\mathrm{dual}}_{\Psi^{*,\tan,2}\mathbf{R}^{\alpha,n}} \left(\Psi^*\sigma, \Psi^*\omega \right) &\approx \mathfrak{T}^{\Psi^*\mathcal{F},\mathrm{dual}}_{\mathbf{R}^{\alpha}_{\Psi}} \left(\Psi^*\sigma, \Psi^*\omega \right). \end{aligned}$$

Now, noting that both of the measures $\widetilde{\sigma}$ and $\widetilde{\omega}$ are compactly supported, we use that both of the error operators $\mathcal{E}^{\alpha}_1 f \widetilde{\sigma}(x) \equiv \int \mathbf{E}^{\alpha}_1(x,y) f(y) d\widetilde{\sigma}(y)$ and $\mathcal{E}^{\alpha}_2 f \widetilde{\omega}(x) \equiv \int \mathbf{E}^{\alpha}_2(x,y) f(y) d\widetilde{\omega}(y)$ have norms controlled by the offset A^{α}_2 condition to obtain

$$\mathfrak{T}^{\Psi^*\mathcal{F}}_{\Psi^{*,\tan,1}\mathbf{R}^{\alpha,n}} \left(\Psi^*\sigma, \Psi^*\omega \right) + \left[\mathcal{A}^{\alpha}_2 + \mathcal{A}^{\alpha,\mathrm{dual}}_2 \right] \approx \mathfrak{T}^{\mathcal{F}}_{\mathbf{R}^{\alpha,n}} (\sigma,\omega) + \left[\mathcal{A}^{\alpha}_2 + \mathcal{A}^{\alpha,\mathrm{dual}}_2 \right],$$

and

$$\mathfrak{T}^{\Psi^*\mathcal{F},\mathrm{dual}}_{\Psi^{*,\tan,2}\mathbf{R}^{\alpha,n}} \left(\Psi^*\sigma, \Psi^*\omega \right) + \left[\mathcal{A}^{\alpha}_2 + \mathcal{A}^{\alpha,\mathrm{dual}}_2 \right] \approx \mathfrak{T}^{\mathcal{F},\mathrm{dual}}_{\mathbf{R}^{\alpha,n}} (\sigma,\omega) + \left[\mathcal{A}^{\alpha}_2 + \mathcal{A}^{\alpha,\mathrm{dual}}_2 \right].$$

Note that once again we need the one-tailed Muckenhoupt conditions to reduce to the case where both measures have common compact support. Combining inequalities we have

$$\begin{aligned} \mathfrak{T}^{\Psi^*\mathcal{F}}_{\mathbf{R}^{\alpha}_{\Psi}} \left(\Psi^*\sigma, \Psi^*\omega \right) + \left[\mathcal{A}^{\alpha}_2 + \mathcal{A}^{\alpha,\mathrm{dual}}_2 \right] &\approx \mathfrak{T}^{\mathcal{F}}_{\mathbf{R}^{\alpha,n}} (\sigma,\omega) + \left[\mathcal{A}^{\alpha}_2 + \mathcal{A}^{\alpha,\mathrm{dual}}_2 \right], \\ \mathfrak{T}^{\Psi^*\mathcal{F},\mathrm{dual}}_{\mathbf{R}^{\alpha}_{\Psi}} \left(\Psi^*\sigma, \Psi^*\omega \right) + \left[\mathcal{A}^{\alpha}_2 + \mathcal{A}^{\alpha,\mathrm{dual}}_2 \right] &\approx \mathfrak{T}^{\mathcal{F},\mathrm{dual}}_{\mathbf{R}^{\alpha,n}} (\sigma,\omega) + \left[\mathcal{A}^{\alpha}_2 + \mathcal{A}^{\alpha,\mathrm{dual}}_2 \right], \end{aligned}$$

and this completes the proof of Proposition 29. □

4.2. **A preliminary $T1$ theorem.** We can use just the change of variable Proposition 29 and Theorem 17 to prove the preliminary Theorem 18. Recall that $\mathcal{L}$ is presented as the graph of a $C^{1,\delta}$ function $\psi : \mathbb{R} \to \mathbb{R}^n$ given by

$$\psi(t) = \left(\psi^2(t), \psi^3(t), ..., \psi^n(t) \right),$$

and that both

$$\begin{aligned} \Psi(x) &= \left(x^1, x^2 - \psi^2\left(x^1\right), x^3 - \psi^3\left(x^1\right), ..., x^n - \psi^n\left(x^1\right) \right) = x - \left(0, \psi\left(x^1\right) \right), \\ \Psi^{-1}(\xi) &= \left(\xi^1, \xi^2 + \psi^2(\xi_1), \xi^3 + \psi^3(\xi_1), ..., \xi^n + \psi^n(\xi_1) \right) = \xi + \psi(\xi_1), \end{aligned}$$

are $C^{1,\delta}$ maps, and that Ψ is a $C^{1,\delta}$ homeomorphism from the curve $\mathcal{L}$ to the x_1-axis. Recall $\Psi_*\mathcal{Q} = \left(\Psi^{-1}\right)^* \mathcal{Q} = \{\Psi Q : Q \in \mathcal{Q}\}$. In the next subsection, the small Lipschitz assumption (1.7) will be removed, and the testing conditions below will be permitted to be taken over usual cubes.

Finally recall that in Theorem 18, we assume the small Lipschitz condition (1.7), i.e.

$$\|D\psi\|_\infty < \frac{1}{8n} \left(1 - \frac{\alpha}{n} \right),$$

and that ω and σ are positive Borel measures (possibly having common point masses) with ω compactly supported in $\mathcal{L}$, and that $\mathcal{R}^n$=$R\mathcal{P}^n$ where R is a fixed rotation that is L-transverse when L is the x_1-axis. The conclusion of Theorem 18 is then that

$$\int_{\Psi Q} \left|\mathbf{R}^{\alpha,n}\left(\mathbf{1}_{\Psi Q}\sigma\right)\right|^2 d\omega \leq \mathfrak{T}^{\Psi\mathcal{Q}}_{\mathbf{R}^{\alpha,n}} \left|\Psi Q\right|_\sigma ,$$

$$\int_{\Psi Q} \left|\mathbf{R}^{\alpha,n,\mathrm{dual}}\left(\mathbf{1}_{\Psi Q}\omega\right)\right|^2 d\sigma \leq \mathfrak{T}^{\Psi\mathcal{Q},\mathrm{dual}}_{\mathbf{R}^{\alpha,n}} \left|\Psi Q\right|_\omega ,$$

$$\text{for all cubes } Q \in \mathcal{R}^n.$$

Proof of Theorem 18. By the testing equivalences (2) and (3) of Proposition 29 with $\mathcal{F} = \Psi\mathcal{Q}$, and using $\Psi^*\mathcal{F}$ =$\Psi^*\Psi\mathcal{Q} = \mathcal{Q}$, we have

$$\begin{aligned} & \sqrt{\mathfrak{A}_2^\alpha\left(\sigma,\omega\right)} + \mathfrak{T}^{\Psi\mathcal{Q}}_{\mathbf{R}^{\alpha,n}}\left(\sigma,\omega\right) + \mathfrak{T}^{\Psi\mathcal{Q},\mathrm{dual}}_{\mathbf{R}^{\alpha,n}}\left(\sigma,\omega\right) \\ \approx\quad & \sqrt{\mathfrak{A}_2^\alpha\left(\sigma,\omega\right)} + \mathfrak{T}^{\mathcal{Q}}_{\mathbf{R}^{\alpha,n}_{\Psi}}\left(\Psi^*\sigma,\Psi^*\omega\right) + \mathfrak{T}^{\mathcal{Q},\mathrm{dual}}_{\mathbf{R}^{\alpha,n}_{\Psi}}\left(\Psi^*\sigma,\Psi^*\omega\right) \\ \approx\quad & \sqrt{\mathfrak{A}_2^\alpha\left(\sigma,\omega\right)} + \mathfrak{N}_{\mathbf{R}^{\alpha,n}_{\Psi}}\left(\Psi^*\sigma,\Psi^*\omega\right), \end{aligned}$$

where the final line follows from Theorem 17 because $\Psi^*\omega$ is supported on a line. Then we continue with equivalence (1) of Proposition 29 to obtain

$$\sqrt{\mathfrak{A}_2^\alpha\left(\sigma,\omega\right)} + \mathfrak{N}_{\mathbf{R}^{\alpha,n}_{\Psi}}\left(\Psi^*\sigma,\Psi^*\omega\right) \approx \sqrt{\mathfrak{A}_2^\alpha\left(\sigma,\omega\right)} + \mathfrak{N}_{\mathbf{R}^{\alpha,n}}\left(\sigma,\omega\right).$$

Altogether we now obtain from this that

$$\mathfrak{N}_{\mathbf{R}^{\alpha,n}}\left(\sigma,\omega\right) \approx \sqrt{\mathfrak{A}_2^\alpha\left(\sigma,\omega\right)} + \mathfrak{T}^{\Psi\mathcal{Q}}_{\mathbf{R}^{\alpha,n}}\left(\sigma,\omega\right) + \mathfrak{T}^{\Psi\mathcal{Q},\mathrm{dual}}_{\mathbf{R}^{\alpha,n}}\left(\sigma,\omega\right),$$

and this completes the proof of Theorem 18. □

Remark 30. *At this point, one can obtain a 'T1 type' theorem when ω is compactly supported on a $C^{1,\delta}$ curve $\mathcal{L}$,* without *any additional restriction on the Lipschitz constant of the curve, by decomposing $\omega = \sum_{i=1}^N \omega_i$ into finitely many measures ω_i with support so small that the supporting curve is presented as a graph of a $C^{1,\delta}$ function ψ_i relative to a rotated axis, and such that $\|D\psi_i\|_\infty < \frac{1}{8n}\left(1-\frac{\alpha}{n}\right)$ (this requires some work). Then Theorem 18 applies to each measure pair (ω_i,σ) (appropriately rotated), and the corresponding rotated quasitesting conditions must now be taken over the finitely many measure pairs (ω_i,σ). In the next subsection we will improve on this observation by eliminating the small Lipschitz assumption, and by taking the testing conditions over the single measure pair (ω,σ).*

4.3. **The $T1$ theorem for a measure supported on a regular $C^{1,\delta}$ curve.** Now we prove our main result, Theorem 9.

Proof of Theorem 9. **Step 1**: Given $0 \leq \alpha < n$, we define

$$\varepsilon \equiv \frac{1}{8n}\left(1-\frac{\alpha}{n}\right),$$

where the right hand side is the constant appearing in (1.7) in Theorem **??**. Now let $0 < \varepsilon' < \varepsilon$ and choose a finite collection of points $\left\{\xi_j\right\}_{j=1}^j \subset \mathbb{S}^{n-1}$ in the unit sphere such that the spherical balls $\left\{\mathcal{B}\left(\xi_j,\frac{\varepsilon'}{4}\right)\right\}_{j=1}^J$ cover $\mathbb{S}^{n-1}$. Observe that our curve Φ and its derivative Φ' are *uniformly* continuous. We now claim that we can

decompose the curve $\mathcal{L}$ into finitely many consecutive pieces $\{\mathcal{L}_i\}_{i=0}^N$ such that with $\widehat{\mathcal{L}_i}$ defined to be $\bigcup_{k=-C_n}^{C_n} \mathcal{L}_{i+k}$, the union of $\mathcal{L}_i$ and the previous and subsequent C_n pieces, and $\mathcal{L}_i^*$ defined to be $\bigcup_{k=-2C_n}^{2C_n} \mathcal{L}_{i+k}$, where $C_n = 5\widetilde{C_n}$ and $\widetilde{C_n}$ is a dimensional constant defined in (4.16) in step 2 below (without circularity), the following three properties hold:

(**1**): there is $\eta > 0$ such that $\mathcal{L}_i = \operatorname{range}\Phi_i$ where $\Phi_i = \Phi \mid_{[\eta i,\eta(i+1))}$ is the restriction of Φ to the interval $[\eta i, \eta(i+1))$ for all i, and

(**2**): for each i, there is $j = j(i)$ depending on i such that, after a rotation R_i that takes the point ξ_j to the point $(0,...,0,1)$, followed by an appropriate translation T_i, the curve $T_i R_i \widehat{\mathcal{L}_i}$ is the graph of the restriction $\psi_i \mid_{[-\zeta_i,\zeta_i)}$ of a globally defined $C^{1,\delta}$ function $\psi_i : \mathbb{R} \to \mathbb{R}^{n-1}$ with $\|D\psi_i\|_\infty < \varepsilon$, and

(**3**): $\psi_i(t) = (0,...,0) \in \mathbb{R}^{n-1}$ for all $|t| > 4\zeta_i$.

Thus we are claiming that we can locally rotate and translate the curve so that it is given locally as part of the graph of a globally defined $C^{1,\delta}$ function ψ_i with $\|D\psi_i\|_\infty < \varepsilon$, where in view of the definition of ε, this latter inequality is what is required in (1.7) of Theorem 18. Note that $\zeta_i \approx C_n\eta$.

To see that these three properties can be obtained, we use uniform continuity of Φ' to take a small piece $\mathcal{L}_i^*$ of the curve, such that the oscillation of tangent lines across the piece $\mathcal{L}_i^*$ is less than ε', and then translate and rotate the chord joining its endpoints so as to lie on the x_1-axis. Note that with this done, the resulting curve is the graph of a function $\psi_i^*(t)$ defined for $t \in I_i^*$, which satisfies

$$|\psi_i^*(t)| \leq C\frac{\varepsilon'}{2}\eta, \qquad t \in I_i^*,$$

since $\psi_i^* = T_i R_i \Phi \mid_{I_i^*}$. Here we are using the convention that I_i is the parameter interval of Φ corresponding to the image $\mathcal{L}_i$, and similarly for $\widehat{I_i}$ and I_i^*.

Then we construct the extended function $\psi_i(t)$ so that its graph includes the translated and rotated piece $\widehat{\mathcal{L}_i}$, and so that away from $\widehat{I_i}$ the function ψ_i smoothly straightens out from ψ_i^* so as to vanish on the remaining x_1-axis, and in such a way that $\|D\psi_i\|_\infty < \varepsilon$. This is most easily seen by taking $\psi_i(t) \equiv \psi_i^*(t)\rho(t)$ where $\rho(t)$ is an appropriate smooth bump function that is identically 1 on $\widehat{I_i}$ and vanishes outside I_i^*. Note then that

$$D\psi_i(t) = D\psi_i^*(t)\rho(t) + \psi_i^*(t) D\rho(t)$$

satisfies $|D\psi_i(t)| \leq C\varepsilon'$ since $|\psi_i^*(t)| \leq C\frac{\varepsilon'}{2}\eta$ and $|D\rho(t)| \leq C\frac{1}{\eta}$. Consequently we have

$$\|D\psi_i\|_\infty \leq C\varepsilon' < \varepsilon.$$

Of course the function $\psi_i = \psi_i^*(t)\rho(t)$ is $C^{1,\delta}$ since $\operatorname{supp}\rho$ is contained in the interior of the interval I_i^*. This completes the verification of properties (1), (2) and (3) above.

In the next step, we will restrict ω to the small piece $\mathcal{L}_i$ and it will be important that we can straighten out the larger piece $\widehat{\mathcal{L}_i}$ via a global $C^{1,\delta}$ diffeomorphism Ψ_i^{-1} of $\mathbb{R}^n$ (defined using ψ_i), so that we can derive a *tripled* quasitesting condition for

intervals in I_i whose triples are contained in $\widehat{I_i}$, the straightened out portion of $\widehat{\mathcal{L}_i}$.

Step 2: We now apply Theorem 13 to the pullbacks $\widehat{I_i}$ under ψ_i of the localized pieces $\widehat{\mathcal{L}_i}$ as follows. Fix i and denote by ω_i the restriction of ω to $\mathcal{L}_i$. We are assuming the usual cube testing conditions $\mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\mathcal{Q}^n}(\sigma,\omega)$ and $\mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\mathcal{Q}^n,\mathrm{dual}}(\sigma,\omega)$ on the weight pair (σ,ω) over all cubes $Q\in\mathcal{Q}^n$. Under the change of variable given by the $C^{1,\delta}$ map $\Psi_i^{-1}:\mathbb{R}^n\to\mathbb{R}^n$, corresponding to $\Psi_i(x)=(x_1,x'+\psi_i(x_1))$, the pair of measures (σ,ω) is transformed to the pullback pair $(\widetilde{\sigma},\widetilde{\omega})$ (since i is fixed we suppress the dependence of the change of variable on i and simply write $\widetilde{\sigma}$ and $\widetilde{\omega}$, but we will use the subscript i to emphasize restrictions of $\widetilde{\omega}$). Now define $\widetilde{\omega_i}$ to be the transform of the small piece of measure ω_i, and note that it is supported on the x_1-axis, and moreover that the transform $\widehat{I_i}$ of the larger piece $\widehat{\mathcal{L}_i}$ is also supported on the x_1-axis. By Proposition 29, and in the presence of $\mathfrak{A}_2^\alpha$, the testing conditions $\mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\mathcal{Q}^n}(\sigma,\omega)$ and $\mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\mathcal{Q}^n,\mathrm{dual}}(\sigma,\omega)$ for the measure pair (σ,ω) over cubes in $\mathcal{Q}^n$ for the α-fractional Riesz transform $\mathbf{R}^{\alpha,n}$ are transformed into the testing conditions $\mathfrak{T}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Psi_i^*\mathcal{Q}^n}(\sigma,\omega)$ and $\mathfrak{T}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Psi_i^*\mathcal{Q}^n,\mathrm{dual}}(\sigma,\omega)$ for the measure pair $(\widetilde{\sigma},\widetilde{\omega})$ over quasicubes in $\Psi_i^*\mathcal{Q}^n$ for the conformal α-fractional Riesz transform $\mathbf{R}_{\Psi}^{\alpha,n}$.

Now in order to apply Theorem 13 to the conformal α-fractional Riesz transform $\mathbf{R}_{\Psi}^{\alpha,n}$, we will choose below a specific rotation $\mathcal{R}^n=R\mathcal{P}^n$ of the collection of cubes $\mathcal{P}^n$. Then we consider the testing conditions for the pair $(\widetilde{\sigma},\widetilde{\omega})$ over these quasicubes $\Psi_i^*\mathcal{R}^n$ that form a subset of the quasicubes $\Psi_i^*\mathcal{Q}^n$. Provided we choose $\varepsilon>0$ small enough in Step 1, the map Ψ_i^* will have its derivative $D\Psi_i^*$ close to the identity I. Choose a rotation R that is L-transverse when L is the x_1-axis. From Lemma 26 applied with $\Omega=\Psi_i^{-1}\circ R$, we then obtain (3.26) and the key geometric property:

(4.15) The intersection of any $Q\in\Psi_i^*\mathcal{R}^n$ with the x_1-axis is an **interval** in the x_1-axis.

Using (3.26) and this geometric property we will now deduce, for the special fractional Riesz transform $\mathbf{R}_{\Psi}^{\alpha,n}$, that in the presence of the $\mathcal{A}_2^\alpha$ conditions, the $\Psi_i^*\mathcal{R}^n$-quasicube testing conditions for the pair $(\widetilde{\sigma},\widetilde{\omega_i})$ follow from the $\Psi_i^*\mathcal{R}^n$-quasicube testing conditions for the pair $(\widetilde{\sigma},\widetilde{\omega})$.

Indeed, fix a quasicube $Q\in\Psi_i^*\mathcal{R}^n$ and consider the left hand sides of the two dual testing conditions, namely

$$\int_Q\left|\mathbf{R}_{\Psi}^{\alpha,n}\mathbf{1}_Q\widetilde{\sigma}\right|^2d\widetilde{\omega_i}\text{ and }\int_Q\left|\mathbf{R}_{\Psi}^{\alpha,n}\mathbf{1}_Q\widetilde{\omega_i}\right|^2d\widetilde{\sigma}.$$

Now the first integral is trivially dominated by $\int_Q\left|\mathbf{R}_{\Psi}^{\alpha,n}\mathbf{1}_Q\widetilde{\sigma}\right|^2d\widetilde{\omega}\le\left(\mathfrak{T}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Psi_i^*\mathcal{R}^n}\right)^2|Q|_{\widetilde{\sigma}}$ as required. To estimate the second integral, we first use (4.15) to choose a quasi cube $Q'\in\Psi^*\mathcal{R}^n$ such that $Q'\subset Q$ and $\mathbf{1}_{Q'}\widetilde{\omega}=\mathbf{1}_Q\widetilde{\omega_i}$, and in addition that

$$\ell(Q')\le\widetilde{C_n}\,\ell(I)\tag{4.16}$$

where $I = Q \cap \{x_1 - axis\}$. This latter condition (4.16) simply means that Q' is taken essentially as small as possible so that $\mathbf{1}_{Q'}\widetilde{\omega} = \mathbf{1}_Q \widetilde{\omega_i}$. Then we write

$$\begin{aligned}
\int_Q \left|\mathbf{R}_{\Psi}^{\alpha,n}\mathbf{1}_Q\widetilde{\omega_i}\right|^2 d\widetilde{\sigma} &= \int_Q \left|\mathbf{R}_{\Psi}^{\alpha,n}\mathbf{1}_{Q'}\widetilde{\omega}\right|^2 d\widetilde{\sigma} \\
&= \int_{Q\setminus 3Q'} \left|\mathbf{R}_{\Psi}^{\alpha,n}\mathbf{1}_{Q'}\widetilde{\omega}\right|^2 d\widetilde{\sigma} + \int_{Q\cap 3Q'} \left|\mathbf{R}_{\Psi}^{\alpha,n}\mathbf{1}_{Q'}\widetilde{\omega}\right|^2 d\widetilde{\sigma} \\
&= I + II.
\end{aligned}$$

Now $I \lesssim \mathcal{A}_2^{\alpha,\mathrm{dual}} \left|Q'\right|_{\widetilde{\omega}} = \mathcal{A}_2^{\alpha,\mathrm{dual}} \left|Q\right|_{\widetilde{\omega_i}}$ by a standard calculation similar to that in (4.9), and

$$II \leq \int_{3Q'} \left|\mathbf{R}_{\Psi}^{\alpha,n}\mathbf{1}_{Q'}\widetilde{\omega}\right|^2 d\widetilde{\sigma} \leq \left(\mathfrak{T}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Psi_i^*\mathcal{R}^n,\mathrm{triple},\mathrm{dual}}\right)^2 \left|Q'\right|_{\widetilde{\omega}} = \left(\mathfrak{T}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Psi_i^*\mathcal{R}^n,\mathrm{triple},\mathrm{dual}}\right)^2 \left|Q\right|_{\widetilde{\omega_i}},$$

by the *local* backward triple quasicube testing condition for the pair $(\widetilde{\sigma}, \widetilde{\omega})$, whose necessity was proved in Theorem 13 above with the measure $\widehat{\widetilde{\omega}}_i$, the restriction of $\widetilde{\omega}$ to $\widehat{I}_i$, which is compactly supported on the real axis. By *local* backward triple quasicube testing here we mean that we are restricting attention to those triples $3Q'$ such that $3Q' \cap \{x_1 - axis\} \subset [-\zeta_i, \zeta_i)$, the image of the larger piece $\widehat{\mathcal{L}_i}$ under ψ_i^{-1}. This restriction is necessary since our arguments for necessity of triple testing require support on a line. Now we use condition (4.16) in our choice of quasicube Q'. Indeed, with this choice we then have $3Q' \cap \{x_1 - axis\} \subset [-\zeta_i, \zeta_i)$, and so have the backward tripled quasicube testing condition at our disposal.

Step 3: Now we use Theorem 13 again to obtain the quasienergy conditions for the conformal fractional Riesz transform $\mathbf{R}_{\Psi}^{\alpha,n}$ for each pair $(\widetilde{\sigma}, \widetilde{\omega_i})$. Thus, *assuming only the $\mathfrak{A}_2^{\alpha}$ conditions, and $\mathcal{Q}^n$ cube testing conditions for the weight pair (σ, ω) for the α-fractional Riesz transform* $\mathbf{R}^{\alpha,n}$, we have established that the weight pair $(\widetilde{\sigma}, \widetilde{\omega_i})$ satisfies the $\mathfrak{A}_2^{\alpha}$ conditions, the quasienergy conditions, the quasitesting conditions and the quasiweak boundedness property (which follows from the backward triple quasitesting condition) all for the conformal α-fractional Riesz transform $\mathbf{R}_{\Psi}^{\alpha,n}$. Now Theorem 5 for conformal Riesz transforms (Conclusion 15) and parts (2) and (3) of Proposition 29 apply to show that

$$\mathfrak{N}_{\mathbf{R}_{\Psi}^{\alpha,n}}\left(\widetilde{\sigma}, \widetilde{\omega_i}\right) \lesssim \sqrt{\mathfrak{A}_2^{\alpha}\left(\sigma,\omega\right)} + \mathfrak{T}_{\mathbf{R}^{\alpha,n}}\left(\sigma,\omega\right) + \mathfrak{T}_{\mathbf{R}^{\alpha,n}}^{\mathrm{dual}}\left(\sigma,\omega\right)$$

for each i. Then by part (1) of Proposition 29 we have

$$\mathfrak{N}_{\mathbf{R}_{\sigma}^{\alpha,n}}\left(\sigma, \omega_i\right) \lesssim \mathfrak{N}_{\mathbf{R}_{\Psi}^{\alpha,n}}\left(\widetilde{\sigma}, \widetilde{\omega_i}\right),$$

and we have

$$\mathfrak{N}_{\mathbf{R}_{\sigma}^{\alpha,n}}\left(\sigma, \omega\right) \leq \sum_{i=1}^{N} \mathfrak{N}_{\mathbf{R}_{\sigma}^{\alpha,n}}\left(\sigma, \omega_i\right).$$

This completes the proof of Theorem 9. □

5. Appendix

Here we state and prove extensions of Theorems 13 and 17 that hold for measures σ and ω supported in a (k_1+1)-plane and (k_2+1)-plane respectively that intersect

in the x_1-axis at right angles. We begin with the following extension of Theorem 13 to perpendicular subspaces.

Theorem 31. *Fix a collection of Ω-quasicubes in $\mathbb{R}^n = \mathbb{R}\times\mathbb{R}^{k_1}\times\mathbb{R}^{k_2}$. Let*

$$\begin{aligned} S &= \left\{(x_1,x',0)\in\mathbb{R}\times\mathbb{R}^{k_1}\times\mathbb{R}^{k_2} : (x_1,x')\in\mathbb{R}\times\mathbb{R}^{k_1}\right\},\\ W &= \left\{(x_1,0,x'')\in\mathbb{R}\times\mathbb{R}^{k_1}\times\mathbb{R}^{k_2} : (x_1,x'')\in\mathbb{R}\times\mathbb{R}^{k_2}\right\},\\ L &= S\cap W = \left\{(x_1,0,0)\in\mathbb{R}\times\mathbb{R}^{k_1}\times\mathbb{R}^{k_2} : x_1\in\mathbb{R}\right\}, \end{aligned}$$

be (k_1+1)-, (k_2+1)- and 1- dimensional subspaces respectively of $\mathbb{R}^n$. Let σ and ω be locally finite positive Borel measures supported on S and W respectively (possibly having common point masses in the intersection L of their supports). Suppose that $\mathbf{R}_{\Psi}^{\alpha,n}$ is a conformal α-fractional Riesz transform with $0\leq\alpha<n$ and graphing function $\Psi(x) = x-(0,\psi(x_1))$ that satisfies (1.7), i.e.

$$\|D\psi\|_\infty < \frac{1}{8n}\left(1-\frac{\alpha}{n}\right), \tag{5.1}$$

and consider the tangent line truncations for $\mathbf{R}_{\Psi}^{\alpha,n}$ in the Ω-quasitesting conditions. Then

$$\mathcal{E}_\alpha^{\Omega\mathcal{Q}^n} \lesssim \sqrt{\mathcal{A}_2^\alpha}+\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n} \text{ and } \mathcal{E}_\alpha^{\Omega\mathcal{Q}^n,\mathrm{dual}} \lesssim \sqrt{\mathcal{A}_2^{\alpha,\mathrm{dual}}}+\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{dual}}.$$

In addition if Ω is a C^1 diffeomorphism that is L-transverse, then

$$\mathcal{WBP}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n} \lesssim \mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{triple},\mathrm{dual}} \lesssim \mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{dual}}+\sqrt{\mathcal{A}_2^\alpha}+\sqrt{\mathcal{A}_2^{\alpha,\mathrm{dual}}}.$$

Proof. In our current situation, the assumptions on σ and ω are symmetric so that it is enough to prove just that the forward quasienergy condition $\mathcal{E}_\alpha^{\Omega\mathcal{Q}^n}$ is bounded by a constant multiple of $\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n}+\sqrt{\mathcal{A}_2^\alpha}$, where $\mathcal{A}_2^\alpha$ is the Muckenhoupt condition with holes. We must show

$$\sup_{\ell\geq0}\sum_{r=1}^\infty\sum_{J\in\mathcal{M}_{\mathrm{deep}}^\ell(I_r)}\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus J^*}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\|\mathsf{P}_J^\omega\mathbf{x}\|_{L^2(\omega)}^2 \lesssim \left(\left(\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n}\right)^2+\mathcal{A}_2^\alpha\right)|I|_\sigma,$$

for all partitions of a dyadic quasicube $I=\dot{\bigcup}_{r\geq1}I_r$ into dyadic subquasicubes I_r. We again fix $\ell\geq0$ and suppress both ℓ and $\mathbf{r}$ in the notation $\mathcal{M}_{\mathrm{deep}}(I_r)=\mathcal{M}_{\mathbf{r}-\mathrm{deep}}^\ell(I_r)$. We may assume that all the quasicubes J intersect $\operatorname{supp}\omega$, since otherwise $\|\mathsf{P}_J^\omega\mathbf{x}\|_{L^2(\omega)}^2=0$, hence that all the quasicubes I_r and J intersect W, which contains $\operatorname{supp}\omega$. Thus what we must show is

$$\sum_{r=1}^\infty\sum_{J\in\mathcal{M}_{\mathrm{deep}}(I_r)}\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus J^*}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\|\mathsf{P}_J^\omega\mathbf{x}\|_{L^2(\omega)}^2 \lesssim \left(\left(\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n}\right)^2+\mathcal{A}_2^\alpha\right)|I|_\sigma, \tag{5.2}$$

where since ω is supported in W,

$$\|\mathsf{P}_J^\omega\mathbf{x}\|_{L^2(\omega)}^2 = \left\|\mathsf{P}_J^\omega x^1\right\|_{L^2(\omega)}^2+\sum_{j=k_1+2}^n\left\|\mathsf{P}_J^\omega x^j\right\|_{L^2(\omega)}^2. \tag{5.3}$$

Let

$$\mathcal{C}(I) \equiv \left\{ J \in \dot{\bigcup_{r\geq 1}} \mathcal{M}_{\text{deep}}(I_r) : J \cap W \neq \emptyset \right\}$$

be the collection of all quasicubes J arising in (5.2). We divide the quasicubes J in $\mathcal{C}(I)$ into two separate collections $\mathcal{C}_i(I)$, $1 \leq i \leq 2$, according to how close J is to the plane S containing the support of σ:

$$\begin{aligned} \mathcal{C}_1(I) &\equiv \{J \in \mathcal{C}(I) : 3J \cap S = \emptyset\}, \\ \mathcal{C}_2(I) &\equiv \{J \in \mathcal{C}(I) : 3J \cap S \neq \emptyset\}. \end{aligned}$$

For the first collection $\mathcal{C}_1(I)$ we estimate the corresponding sum in (5.2) using weak reversal of energy. Indeed, for $J \in \mathcal{C}_1(I)$ we use the argument in (3.12) and (3.13), but with σ and ω interchanged, to see that for any $x \in J$,

$$\begin{aligned} &\left(\frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_{I\setminus J^*}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \|\mathsf{P}_J^\omega \mathbf{x}\|_{L^2(\omega)}^2 \\ \lesssim\ &\left(\frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_{I\setminus J^*}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 |J|^{\frac{2}{n}} |J|_\omega = \mathrm{P}^\alpha\left(J, \mathbf{1}_{I\setminus J^*}\sigma\right)^2 |J|_\omega \leq \mathrm{P}^\alpha\left(J, \mathbf{1}_I\sigma\right)^2 |J|_\omega \\ \lesssim\ &\left|\mathbf{R}_\Psi^{\alpha,n}\left(\mathbf{1}_I\sigma\right)(x)\right|^2 |J|_\omega\ , \end{aligned}$$

and so using that the $J \in \mathcal{C}(I)$ are pairwise disjoint and contained in I, we get

$$\begin{aligned} \sum_{J\in\mathcal{C}_1(I)} \left(\frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_{I\setminus J^*}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \|\mathsf{P}_J^\omega \mathbf{x}\|_{L^2(\omega)}^2 &\lesssim \sum_{J\in\mathcal{C}_1(I)} \inf_{x\in J} \left|\mathbf{R}_\Psi^{\alpha,n}\left(\mathbf{1}_I\sigma\right)(x)\right|^2\ |J|_\omega \\ &\leq \int_I \left|\mathbf{R}_\Psi^{\alpha,n}\left(\mathbf{1}_I\sigma\right)(x)\right|^2 d\omega(x) \leq \left(\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n}\right)^2 |I|_\sigma\ . \end{aligned}$$

Now we consider the quasicubes J in the second collection $\mathcal{C}_2(I)$. In this case we apply the arguments used above for the forward quasienergy condition in Subsection 3.2. Given $J \in \mathcal{C}_2(I)$ we consider the decomposition

$$I \setminus J^* = \mathrm{E}(J^*)\, \dot{\cup} \mathrm{S}(J^*)$$

of $I \setminus J^*$ into *end* $\mathrm{E}(J^*)$ and *side* $\mathrm{S}(J^*)$ disjoint pieces defined by

$$\begin{aligned} \mathrm{E}(J^*) &\equiv (I \setminus J^*) \cap \left\{ \left(y^1, y'''\right) : |y''' - c_J'''| \leq \frac{10}{\gamma}\left|y^1 - c_J^1\right| \right\}; \\ \mathrm{S}(J^*) &\equiv (I \setminus J^*) \setminus \mathrm{E}(J^*)\ , \end{aligned}$$

where we write $y = \left(y^1, y', y''\right) = \left(y^1, y'''\right)$ with $y''' = (y', y'') \in \mathbb{R}^{k_1+k_2}$. Then by (5.3) it suffices to show

$$\begin{aligned} A^j &\equiv \sum_{J\in\mathcal{C}_2(I)} \left(\frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_{\mathrm{E}(J^*)}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}_J^\omega x^j\right\|_{L^2(\omega)}^2 \lesssim \left(\left(\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n}\right)^2 + \mathcal{A}_2^\alpha\right) |I|_\sigma\ , \\ B &\equiv \sum_{J\in\mathcal{C}_2(I)} \left(\frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_{\mathrm{S}(J^*)}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \|\mathsf{P}_J^\omega \mathbf{x}\|_{L^2(\omega)}^2 \lesssim \left(\left(\mathfrak{T}_{\mathbf{R}_\Psi^{\alpha,n}}^{\Omega\mathcal{Q}^n}\right)^2 + \mathcal{A}_2^\alpha\right) |I|_\sigma\ , \end{aligned}$$

for $j = 1$ and $k_1 + 2 \leq j \leq n$.

We estimate A^1 involving the ends $\mathrm{E}(J^*)$ as before, obtaining first a strong reversal of the x^1-energy $\left\|\mathsf{P}_J^{\omega}x^1\right\|_{L^2(\omega)}^2$, followed by an '$\mathcal{A}_2^{\alpha}$ reversal' of the other partial energies $\left\|\mathsf{P}_J^{\omega}x^j\right\|_{L^2(\omega)}^2$ for $k_1+2\leq j\leq n$. In particular, the strong reversal of x^1-energy that we obtain here is analogous to that appearing in (3.14), and to that appearing in (3.11) with σ and ω interchanged, and it then delivers the following estimate analogous to (3.17),

$$
\begin{aligned}
&A^1 \equiv \sum_{J\in\mathcal{C}_2(I)}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{\mathrm{E}(J^*)}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\int_{J\cap L}\left|x^1-\mathbb{E}_J^{\omega}x^1\right|^2 d\omega\left(x\right)\\
\lesssim\ & \sum_{J\in\mathcal{C}_2(I)}\frac{1}{|J|_{\omega}}\int_{J\cap L}\int_{J\cap L}\left\{\left(\mathbf{R}_{\Psi}^{\alpha,n}\right)_1\left(\mathbf{1}_I\sigma\right)\left(x^1,0',x''\right)-\left(\mathbf{R}_{\Psi}^{\alpha,n}\right)_1\left(\mathbf{1}_I\sigma\right)\left(z^1,0',z''\right)\right\}^2 d\omega\left(x\right)d\omega\left(z\right)\\
&+\sum_{J\in\mathcal{C}_2(I)}\frac{1}{|J|_{\omega}}\int_{J\cap L}\int_{J\cap L}\left\{\left(\mathbf{R}_{\Psi}^{\alpha,n}\right)_1\left(\mathbf{1}_{J^*}\sigma\right)\left(x^1,0',x''\right)-\left(\mathbf{R}_{\Psi}^{\alpha,n}\right)_1\left(\mathbf{1}_{J^*}\sigma\right)\left(z^1,0',z''\right)\right\}^2 d\omega\left(x\right)d\omega\left(z\right)\\
&+\sum_{J\in\mathcal{C}_2(I)}\frac{1}{|J|_{\omega}}\int_{J\cap L}\int_{J\cap L}\left\{\left(\mathbf{R}_{\Psi}^{\alpha,n}\right)_1\left(\mathbf{1}_{\mathrm{S}(J^*)}\sigma\right)\left(x^1,0',x''\right)-\left(\mathbf{R}_{\Psi}^{\alpha,n}\right)_1\left(\mathbf{1}_{\mathrm{S}(J^*)}\sigma\right)\left(z^1,0',z''\right)\right\}^2 d\omega\left(x\right)d\omega\left(z\right)\\
\equiv\ & A_1^1+A_2^1+A_3^1,
\end{aligned}
$$

where we have used the 'paraproduct trick' $I=J^*\dot{\cup}\left(I\setminus J^*\right)=J^*\dot{\cup}\mathrm{E}(J^*)\,\dot{\cup}\mathrm{S}(J^*)$.

Now we can discard the differences in both of the terms A_1^1 and A_2^1 and use pairwise disjointedness of J and bounded overlap of J^* to control each of A_1^1 and A_2^1 by $\left(\mathfrak{T}_{\left(\mathbf{R}_{\Psi}^{\alpha,n}\right)_1}^{\Omega\mathcal{Q}^n}\right)^2|I|_{\sigma}$. Just as in the previous argument, term A_3^1 is dominated by term B.

Now for $k_1+2\leq j\leq n$ we again use the 'paraproduct trick' $I\setminus J^*=\mathrm{E}(J^*)\,\dot{\cup}\mathrm{S}(J^*)$ to dominate

$$
A^j\equiv\sum_{J\in\mathcal{C}_2(I)}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{\mathrm{E}(J^*)}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\int_{J\cap L}\left|x^j-\mathbb{E}_J^{\omega}x^j\right|^2 d\omega\left(x\right)
$$

by

$$
\begin{aligned}
&\sum_{J\in\mathcal{C}_2(I)}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{I\setminus J^*}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\int_{J\cap L}\left|x^j-\mathbb{E}_J^{\omega}x^j\right|^2 d\omega\left(x\right)\\
&+\sum_{J\in\mathcal{C}_2(I)}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{\mathrm{S}(J^*)}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\int_{J\cap L}\left|x^j-\mathbb{E}_J^{\omega}x^j\right|^2 d\omega\left(x\right)\\
\equiv\ & A_1^j+A_2^j,
\end{aligned}
$$

where, since the directions of x^j are perpendicular to the support of σ for $k_1+2\leq j\leq n$, the term A_1^j is treated using 'weak energy reversal'. Namely, for $x\in J\cap L$,

we have as in (3.12) that,

$$\begin{aligned}
\frac{|x_j|}{|J|^{\frac{1}{n}}}\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus J^*}\sigma\right) &\lesssim \frac{|x_j|}{|J|^{\frac{1}{n}}}\int_{I\setminus J^*}\frac{|J|^{\frac{1}{n}}}{\left(|J|^{\frac{1}{n}}+|y-x|\right)^{n+1-\alpha}}d\sigma(y)\\
&\approx \int_{I\setminus J^*}\frac{|x_j|}{|\Psi(y)-\Psi(x)|^{n+1-\alpha}}d\sigma(y)\\
&\approx \left|\left(\mathbf{R}_\Psi^{\alpha,n}\right)_j\left(\mathbf{1}_{I\setminus J^*}\sigma\right)(x)\right|\lesssim\left|\mathbf{R}_\Psi^{\alpha,n}\left(\mathbf{1}_I\sigma\right)(x)\right|+\left|\mathbf{R}_\Psi^{\alpha,n}\left(\mathbf{1}_{J^*}\sigma\right)(x)\right|,
\end{aligned}$$

which gives

$$\begin{aligned}
A_1^j &= \sum_{J\in\mathcal{C}_2(I)}\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus J^*}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\int_{J\cap L}\left|x^j-\mathbb{E}_J^\omega x^j\right|^2 d\omega(x)\\
&\leq \sum_{J\in\mathcal{C}_2(I)}\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus J^*}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\int_{J\cap L}\left|x^j\right|^2 d\omega(x)\\
&= \sum_{J\in\mathcal{C}_2(I)}\int_{J\cap L}\left(\frac{|x^j|}{|J|^{\frac{1}{n}}}\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus J^*}\sigma\right)\right)^2 d\omega(x)\\
&\lesssim \sum_{J\in\mathcal{C}_2(I)}\int_{J\cap L}\left(\left|\mathbf{R}_\Psi^{\alpha,n}\left(\mathbf{1}_I\sigma\right)\right|^2+\left|\mathbf{R}_\Psi^{\alpha,n}\left(\mathbf{1}_{J^*}\sigma\right)\right|^2\right)d\omega(x),
\end{aligned}$$

and we now proceed as in (3.18) and (3.19). Then we dominate the second term A_2^j by term B.

Now we treat term B using the 'Carleson shadow method' that was used to treat term B in (3.20) in Subsection 3.2. We first assume that $n-1\leq\alpha<n$ so that $\mathrm{P}^\alpha\left(J,\mathbf{1}_{\mathrm{S}(J^*)}\sigma\right)\leq\mathcal{P}^\alpha\left(J,\mathbf{1}_{\mathrm{S}(J^*)}\sigma\right)$, and then use $\|\mathsf{P}_J^\omega\mathbf{x}\|_{L^2(\omega)}^2\lesssim|J|^{\frac{2}{n}}|J|_\omega$ and apply the $\mathcal{A}_2^\alpha$ condition with holes to obtain the following '$\mathcal{A}_2^\alpha$ reversal' of quasienergy,

$$B\lesssim\mathcal{A}_2^\alpha\int_I\left\{\sum_{J\in\mathcal{C}_2(I)}\left(\frac{|J|^{\frac{1}{n}}}{|J|^{\frac{1}{n}}+|y-c_J|}\right)^{n+1-\alpha}\mathbf{1}_{\mathrm{S}(J^*)}(y)\right\}d\sigma(y)\equiv\mathcal{A}_2^\alpha\int_I F(y)\,d\sigma(y).$$

At this point we claim as above that $F(y)\leq C$ with a constant C independent of the decomposition $\mathcal{C}_2(I)$. For this we now define $\mathrm{Sh}(y;\gamma)$ to be the Carleson shadow of the point y onto the x_1-axis L with sides of slope $\frac{10}{\gamma}$, i.e. $\mathrm{Sh}(y;\gamma)$ is the interval on L with length $\frac{1}{5}\gamma\,\mathrm{dist}(y,L)$ and center equal to the point on W that is closest to y. We then proceed as above to the following variant of a previous estimate, where we here redefine $\mathcal{J}\equiv 3J\cap L$ for $J\in\mathcal{C}_2(I)$:

$$\begin{aligned}
F(y) &= \sum_{\substack{J\in\mathcal{C}_2(I)\\ \mathcal{J}\subset C\,\mathrm{Sh}(y;\gamma)}}\left(\frac{|J|^{\frac{1}{n}}}{|J|^{\frac{1}{n}}+|y-c_J|}\right)^{n+1-\alpha}\mathbf{1}_{\mathrm{S}(J^*)}(y)\\
&\lesssim \sum_{r=1}^\infty\sum_{\substack{J\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}(I_r)\\ \emptyset\neq\mathcal{J}\subset C\,\mathrm{Sh}(y;\gamma)}}\left(\frac{|J|^{\frac{1}{n}}}{|y-c_J|}\right)^{n-\alpha}\frac{|J|^{\frac{1}{n}}}{\mathrm{dist}(y,L)}\mathbf{1}_{\mathrm{S}(J^*)}(y),
\end{aligned}$$

and then following the argument for (3.23), we can dominate this by

$$\begin{aligned}\frac{1}{\operatorname{dist}(y,L)}\sum_{r=1}^{\infty}\left\{\sum_{\substack{J\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}(I_r)\\ \emptyset\neq\mathcal{J}\subset C\,\mathrm{Sh}(y;\gamma)}}|J|^{\frac{1}{n}}\right\} &\lesssim \frac{1}{\operatorname{dist}(y,L)}\sum_{r=1}^{\infty}\beta\left|I_r\cap C'\,\mathrm{Sh}(y;\gamma)\right| \\ &\lesssim \beta\frac{1}{\operatorname{dist}(y,L)}\left|C'\,\mathrm{Sh}(y;\gamma)\right|\lesssim\beta\gamma,\end{aligned}$$

since the quasicubes J^* have overlap bounded by β (so that we can essentially treat the shadows as being pairwise disjoint). This completes the proof that

$$B\lesssim\mathcal{A}_2^{\alpha}\int_I F(y)\,d\sigma(y)\lesssim\mathcal{A}_2^{\alpha}\,|I|_{\sigma}\ .$$

Finally, note that the case $0\leq\alpha<n-1$ is handled using the Cauchy-Schwartz inequality as in (3.24).

Remark: Our decomposition into end and side pieces here uses the line L as the means of definition, rather than the possibly larger subspace W, in order to exploit one-dimensional reversal of energy. Of course in the argument for the forward quasienergy condition in Subsection 3.2, the spaces W and L coincide.

Finally, we prove the estimate for the tripled testing condition $\mathfrak{T}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{triple},\mathrm{dual}}$ by exactly the same method as used before in Subsection 3.3. Indeed, with notation analogous to that in Subsection 3.3, we take absolute values inside the fractional singular integral,

$$\int_Q\left|\mathbf{R}_{\Psi}^{\alpha,n}(1_{Q'}\omega)\right|^2 d\sigma\lesssim\int_Q\left\{\int_{Q'}|y-x|^{\alpha-n}\,d\omega(x)\right\}^2 d\sigma(y),$$

and then decompose the two perpendicular sets $Q'\cap W$ and $Q\cap S$ in annuli away from their point of intersection $P\equiv Q'\cap Q\cap L$. Then using that Ω is a C^1 diffeomorphism and L-transverse, and that Q and Q' are neighbouring Ω-quasicubes, the Hardy operator applies just as before in Subsection 3.3. $\square$

It is now an easy matter to obtain from Theorems 31 and 5 the following $T1$ theorem that generalizes Theorem 17.

Theorem 32. *Let*

$$\begin{aligned}S &= \left\{(x_1,x',0)\in\mathbb{R}\times\mathbb{R}^{k_1}\times\mathbb{R}^{k_2}:(x_1,x')\in\mathbb{R}\times\mathbb{R}^{k_1}\right\},\\ W &= \left\{(x_1,0,x'')\in\mathbb{R}\times\mathbb{R}^{k_1}\times\mathbb{R}^{k_2}:(x_1,x'')\in\mathbb{R}\times\mathbb{R}^{k_2}\right\},\\ L &= S\cap W=\left\{(x_1,0,0)\in\mathbb{R}\times\mathbb{R}^{k_1}\times\mathbb{R}^{k_2}:x_1\in\mathbb{R}\right\},\end{aligned}$$

be (k_1+1)-, (k_2+1)- and 1- dimensional subspaces respectively of $\mathbb{R}^n=\mathbb{R}\times\mathbb{R}^{k_1}\times\mathbb{R}^{k_2}$. Let σ and ω be locally finite positive Borel measures supported on S and W respectively (possibly having common point masses in the intersection L of their supports). Suppose that Ω is a C^1 diffeomorphism and L-transverse. Suppose also that $\mathbf{R}_{\Psi}^{\alpha,n}$ is a conformal fractional Riesz transform with $0\leq\alpha<n$, where Ψ is a $C^{1,\delta}$ diffeomorphism given by $\Psi(x)=x-(0,\psi(x_1))$ where ψ satisfies (1.7). Set $\left(\mathbf{R}_{\Psi}^{\alpha,n}\right)_{\sigma}f=\mathbf{R}_{\Psi}^{\alpha,n}(f\sigma)$ for any smooth truncation of $\mathbf{R}_{\Psi}^{\alpha,n}$. Then the operator

norm $\mathfrak{N}_{\mathbf{R}_{\Psi}^{\alpha,n}}$ *of* $\left(\mathbf{R}_{\Psi}^{\alpha,n}\right)_{\sigma}$ *as an operator from* $L^2(\sigma)$ *to* $L^2(\omega)$*, uniformly in smooth truncations, satisfies*

$$\mathfrak{N}_{\mathbf{R}_{\Psi}^{\alpha,n}} \approx C_{\alpha}\left(\sqrt{\mathfrak{A}_2^{\alpha}} + \mathfrak{T}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Omega\mathcal{Q}^n} + \mathfrak{T}_{\mathbf{R}_{\Psi}^{\alpha,n}}^{\Omega\mathcal{Q}^n,\mathrm{dual}}\right).$$

Remark 33. *The above theorem generalizes Theorem 17 by permitting the support of the measure* ω *to extend into an othogonal subspace in a higher dimension. There is an analogous theorem that generalizes Theorem 9 in this way, but we will not pursue this here.*

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